

# Campus Academic Resource Program

## Chain Rule

This handout will:

- Provide a strategy to identify composite functions
- Provide a strategy to find chain rule by using a substitution method.

### Identifying Composite Functions

*This section will provide a strategy to identify composite functions*

**The chain rule** is used to the derivatives of **composite functions**. If you are unfamiliar with the definition of composite functions, please refer here: [Composition of Functions](#). Before using the chain rule, must be able to identify composite functions. The chain rule is defined in the following section.

To identify if a function is a composite functions, you must examine the **argument**, or input, of a function. For example, the argument of the sine function in  $\sin x$  is  $x$ , the argument of the square root function in  $\sqrt{1 + x^2}$  is  $1 + x^2$ , and the argument of the exponential function in  $e^{-x}$  is  $-x$ . **Composite function** always involve a function whose argument is not only  $x$ . For example:

- $y = \cos(\ln x)$  is a composite function since the argument of the cosine function isn't just  $x$ —it's  $\ln x$ . Therefore, you *will* need to use the chain rule to **differentiate**<sup>1</sup> this function.
- $y = 1 + e^{-x^2}$  is a composite function since the argument of the exponential function isn't just  $x$ , it's  $x^2$ . Therefore, you *will* need to use the chain rule to differentiate this function.
- $y = (\ln x) \cdot (\sin x) \cdot (\sqrt{x}) \cdot (e^x)$  is *not* a composite function, since the argument of each function (logarithm, sine, square root, and exponential) is just  $x$ . Rather, the function is a product of four non-composite functions. Therefore, you *do not* need to use the chain rule to differentiate this function.

Conclusion: if you see an argument more complicated than  $x$ , then the function is composite. Because the function is composite, you must use the chain rule to differentiate that function.

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<sup>1</sup> Differentiate is the verb for "finding derivative of." I.e. the verb corresponding to derivative is differentiate, not derive.

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This section will introduce the chain rule and provide 3 worked examples showing how to calculate derivatives that require use of chain rule.

Let  $y = f(x)$  be a composite function. Furthermore, suppose  $y = f(u)$  where  $u$  is some function of  $x$ . Then the **chain rule states**:

$\frac{dy}{dx} = \frac{df(u)}{du} \frac{du}{dx}$	<b>Chain Rule</b>
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Verbally, the chain rule can be said as “the derivative of a composite function is equal to the derivative of a function with respect to its own argument times the derivative of its argument.”

We will first do an example where  $u(x)$  is given; typically, however, you must identify  $u(x)$  yourself—a strategy for this is provided after the first example.

**Example 1:** Suppose  $y = f(x)$ , where  $f(u) = \sin u$  and  $u(x) = x^2 + 2x$ . What is the derivative  $\frac{dy}{dx}$ ? Write your answer as a function of  $x$ .

**Solution:** By the chain rule,

$$\frac{dy}{dx} = \frac{df(u)}{du} \cdot \frac{du}{dx}$$

By using the derivative of sine is cosine

$$\frac{df(u)}{du} = \cos u.$$

by power rule and sum/difference rule.

$$\frac{du}{dx} = 2x + 2.$$

By chain rule

$$\frac{dy}{dx} = (2x + 2) \cos u.$$

However, the problem requires we write our answer as a function of  $x$ . To do so, we substitute in  $u(x) = x^2 + 2x$  to find

$$\frac{dy}{dx} = (2x + 2) \cos(x^2 + 2x).$$

Example 1 demonstrates how a problem is calculated once  $u(x)$  is known. Earlier, we discussed that you may identify a composite function if the argument of a function is more complicated than  $x$ . The argument is what we set to  $u(x)$ . For example:

- If  $y = \cos(x^2 + 1)$ , you set the argument of the cosine function,  $x^2 + 1$ , equal to  $u(x)$ . So  $u = x^2 + 1$ .

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- If  $y = \ln(\sqrt{x})$ , you set the argument of the logarithm function,  $\sqrt{x}$ , equal to  $u(x)$ . So  $u(x) = \sqrt{x}$ ,

Once you have identified  $u(x)$ , you follow the procedure in example 1 to compute the derivative.

**Example 2:** Find the derivative of  $y = \sin(x^2 + 2x)$ .

**Solution:** We may identify  $y = \sin(x^2 + 2x)$  as a composite function by examining the argument—the argument of the sine function is  $x^2 + 2x$ . When we encounter a complicated argument, we set that argument equal to  $u(x)$ .

We are now tasked with finding the derivative of  $y = \sin(u)$  where  $u(x) = x^2 + 2x$ . This was the exact problem solved in Example 1; by following the steps in Example 1 you will find the derivative is  $(2x + 2) \cos(x^2 + 2x)$ .

In Example 2, only one substitution,  $u(x) = x^2 + 2x$ , was necessary to find the derivative. Sometimes, multiple substitutions are necessary. Suppose  $y$  is a function of  $u$ ,  $u$  a function of  $v$ , and  $v$  a function of  $x$ , then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dv} \frac{dv}{dx}$$

**Example 3:** Find the derivative of  $y = \ln(\sqrt{1 + \cos x})$ .

**Solution:** The logarithmic function is a composite function, since its argument is  $\sqrt{1 + \cos x}$ , the first substitution we make is  $u(x) = \sqrt{1 + \cos x}$ . The square root function is a composite function, since its argument is  $1 + \cos x$ . The second substitution we make is  $v(x) = 1 + \cos x$ . Therefore:

$$\begin{aligned}y &= \ln u \\u(v) &= \sqrt{v} \\v &= 1 + \cos x.\end{aligned}$$

We may now use the substitution rule

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dv} \frac{dv}{dx} \\ \frac{dy}{dx} &= \frac{d \ln u}{du} \frac{d \sqrt{v}}{dv} \frac{d(1 + \cos x)}{dx} \\ &= \frac{1}{u} \frac{1}{2\sqrt{v}} (-\sin x)\end{aligned}$$

We must substitute in  $u$  and  $v$  back into  $\frac{dy}{dx}$  until we have  $\frac{dy}{dx}$  only in terms of  $x$ . First, we substitute  $u = \sqrt{v}$  into our expression for  $\frac{dy}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sqrt{v}} \frac{1}{2\sqrt{v}} (-\sin x) \\ &= -\frac{\sin x}{2v}\end{aligned}$$

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Next we substitute  $v(x) = 1 + \cos x$  into our expression for  $\frac{dy}{dx}$ :

$$\frac{dy}{dx} = -\frac{\sin x}{2(1 + \cos x)}$$

which is the derivative of  $y = \ln(\sqrt{1 + \cos x})$ .

#### Glossary:

*This section will define key terms used in this handout.*

**Argument:** The argument is the input of a function.

**Composite Function:** A composite function is a function whose input is another function. Composite functions have an argument that's more complicated than  $x$ .

**Chain Rule:** Rule for finding derivatives of composite function. See page 1 for formula.

**Differentiate:** Differentiate the verb meaning "find derivative of."

#### Practice Problems:

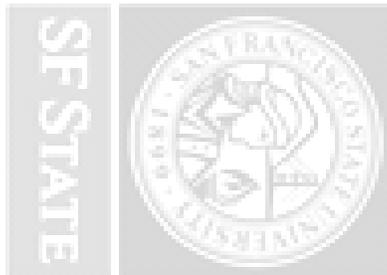
*This section will provide four practice problems.*

1.  $y = (2x + 1)^2$

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2.  $y = (5 - 3x)^{-4} + \frac{1}{8}\left(\frac{2}{3x} + 1\right)^4$



3.  $y = \cos^2 x^2$  (*hint:  $\cos^2 \theta = (\cos \theta)^2$* )

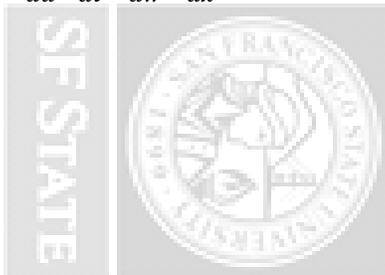
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4.  $y = \cos^3(\sec^3 2t)$  (Hint:  $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}$ )



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- Answers: 1.)  $\frac{dy}{dx} = 8x + 4$   
 2.)  $\frac{dy}{dx} = 12(5 - 3x)^{-5} - \frac{1}{3x^2} \left( \frac{3x}{2} + 1 \right)^3$   
 3.)  $-4x \sin(x^2) \cos(x^2)$   
 4.)  $\frac{dy}{dx} = (3 \cos^2(\sec^3 2t)) \cdot (-\sin(\sec^3 2t)) \cdot (3 \sec^2 2t) \cdot (\tan 2t \sec 2t) \cdot 2$