Campus Academic Resource Program

Chain Rule

This handout will:

- Provide a strategy to identify composite functions
- Provide a strategy to find chain rule by using a substitution method.

Identifying Composite Functions

This section will provide a strategy to identify composite functions

The chain rule is used to the derivatives of **composite functions.** If you are unfamiliar with the definition of composite functions, please refer here: [Composition of Functions](). Before using the chain rule, must be able to identify composite functions. The chain rule is defined in the following section.

To identify if a function is a composite functions, you must examine the **argument,** or input, of a function. For example, the argument of the sine function in $\sin x$ is x, the argument of the square root function in $\sqrt{1+x^2}$ is $1+x^2$, and the argument of the exponential function in e^{-x} is $-x$. **Composite function** always involve a function whose argument is not only x. For example:

- $y = \cos(\ln x)$ is a composite function since the argument of the cosine function isn't just x—it's $\ln x$. Therefore, you *will* need to use the chain rule to **differentiate**[1] this function.
- $y = 1 + e^{-x^2}$ is a composite function since the argument of the exponential function isn't just x, it's x^2. Therefore, you *will* need to use the chain rule to differentiate this function.
- $y = (\ln x) \cdot (\sin x) \cdot (\sqrt{x}) \cdot (e^x)$ is *not* a composite function, since the argument of each function (logarithm, sine, square root, and exponential) is just x. Rather, the function is a product of four non-composite functions. Therefore, you *do not* need to use the chain rule to differentiate this function.

Conclusion: if you see an argument more complicated than x, then the function is composite. Because the function is composite, you must use the chain rule to differentiate that function.

[1] Differentiate is the verb for "finding derivative of." I.e. the verb corresponding to derivative is differentiate, not derive.

Chain Rule

This section will introduce the chain rule and provide 3 worked examples showing how to calculate derivatives that require use of chain rule.

Let $y = f(x)$ be a composite function. Furthermore, suppose $y = f(u)$ where u is some function of x. Then the **chain rule states:**

$\frac{dy}{dx} = \frac{df(u)}{du}\frac{du}{dx}$	**Chain Rule**

Verbally, the chain rule can be said as "the derivative of a composite function is equal to the derivative of a function with respect to its own argument times the derivative of its argument."

We will first do an example where $u(x)$ is given; typically, however, you must identify $u(x)$ yourself—a strategy for this is provided after the first example.

Example 1: Suppose $y = f(x)$, where $f(u) = \sin u$ and $u(x) = x^2 + 2x$. What is the derivative $\frac{dy}{dx}$? Write your answer as a function of x.

Solution: By the chain rule,

$$\frac{dy}{dx} = \frac{df(u)}{du} \cdot \frac{du}{dx}.$$

By using the derivative of sine is cosine

$$\frac{df(u)}{du} = \cos u.$$

by power rule and sum/difference rule.

$$\frac{du}{dx} = 2x + 2.$$

By chain rule

$$\frac{dy}{dx} = (2x + 2)\cos u.$$

However, the problem requires we write our answer as a function of x. To do so, we substitute in $u(x) = x^2 + 2x$ to find

$$\frac{dy}{dx} = (2x + 2)\cos(x^2 + 2x).$$

Example 1 demonstrates how a problem is calculated once $u(x)$ is known. Earlier, we discussed that you may identify a composite function if the argument of a function is more complicated than x. The argument is what we set to $u(x)$. For example:

- If $y = \cos(x^2 + 1)$, you set the argument of the cosine function, $x^2 + 1$, equal to $u(x)$. So $u = x^2 + 1$.

- If $y = \ln(\sqrt{x})$, you set the argument of the logarithm function, $\sqrt{x}$, equal to $u(x)$. So $u(x) = \sqrt{x}$,

Once you have identified $u(x)$, you follow the procedure in example 1 to compute the derivative.

Example 2: Find the derivative of $y = \sin(x^2 + 2x)$.

Solution: We may identify $y = \sin(x^2 + 2x)$ as a composite function by examining the argument—the argument of the sine function is $x^2 + 2x$. When we encounter a complicated argument, we set that argument equal to $u(x)$.

We are now tasked with finding the derivative of $y = \sin(u)$ where $u(x) = x^2 + 2x$. This was the exact problem solved in Example 1; by following the steps in Example 1 you will find the derivative is $(2x + 2)\cos(x^2 + 2x)$.

In Example 2, only one substitution, $u(x) = x^2 + 2x$, was necessary to find the derivative. Sometimes, multiple substitutions are necessary. Suppose y is a function of u, u a function of v, and v a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}.$$

Example 3: Find the derivative of $y = \ln\left(\sqrt{1 + \cos x}\right)$.

Solution: The logarithmic function is a composite function, since its argument is $\sqrt{1 + \cos x}$, the first substitution we make is $u(x) = \sqrt{1 + \cos x}$. The square root function is a composite function, since its argument is $1 + \cos x$. The second substitution we make is $v(x) = 1 + \cos x$. Therefore:

$$y = \ln u$$
$$u(v) = \sqrt{v}$$
$$v = 1 + \cos x.$$

We may now use the substitution rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$$
$$\frac{dy}{dx} = \frac{d\ln u}{du}\,\frac{d\sqrt{v}}{dv}\,\frac{d(1 + \cos x)}{dx}$$
$$= \frac{1}{u}\,\frac{1}{2\sqrt{v}}\,(-\sin x)$$

We must substitute in u and v back into $\frac{dy}{dx}$ until we have $\frac{dy}{dx}$ only in terms of x. First, we substitute $u = \sqrt{v}$ into our expression for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{v}}\,\frac{1}{2\sqrt{v}}\,(-\sin x)$$
$$= -\frac{\sin x}{2v}$$

Next we substitute $v(x) = 1 + \cos x$ into our expression for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{\sin x}{2(1 + \cos x)},$$

which is the derivative of $y = \ln\left(\sqrt{1 + \cos x}\right)$.

Glossary:

This section will define key terms used in this handout.

Argument: The argument is the input of a function.

Composite Function: A composite function is a function whose input is another function. Composite functions have an argument that's more complicated than x.

Chain Rule: Rule for finding derivatives of composite function. See page 1 for formula.

Differentiate: Differentiate the verb meaning "find derivative of."

Practice Problems:

This section will provide four practice problems.

1. $\boldsymbol{y = (2x + 1)^2}$

2. $\boldsymbol{y = (5 - 3x)^{-4} + \frac{1}{8}\left(\frac{2}{3x} + 1\right)^{4}}$

3. $\boldsymbol{y = \cos^2 x^2}$ *(hint:* $\cos^2\theta = (\cos\theta)^2$*)*

4. $y = \cos^3(\sec^3 2t)$ (*Hint*: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}$)

4.) $\frac{dy}{dx} = (3\cos^2(\sec^3 2t)) \cdot (-\sin(\sec^3 2t)) \cdot (3\sec^2 2t)(\tan 2t \sec 2t) \cdot 2$

Answers: 1.) $\frac{dy}{dx} = 8x + 4$ 2.) $\frac{dy}{dx} = 12(5-3x)^{-5} - \frac{1}{3x^2}\left(\frac{2}{3x}+1\right)^3$ 3.) $-4x\sin(x^2)\cos(x^2)$