Campus Academic Resource Program

Differentiation Rules

This handout **will**:

- Outline the definition of the derivative and introduce different notations for the derivative
- Introduce differentiation rules and provide concise explanation of them
- Provide examples of applications of differentiation rules

This handout **will not** discuss:

- The Chain Rule
  - Information on chain rule may be found here: Chain Rule
- Trigonometric Rules
- Logarithmic Rule

Overview of Derivatives and Derivative notations

(For information on how the derivative is defined in terms of a limit, refer to the limit definition of the derivative handout: http://carp.sfsu.edu/sites/sites7.sfsu.edu.carp/files/The Limit Definition of the Derivative.pdf)

This section will:

- Overview the definition of the derivative
- Overview of different notations for derivatives

Derivative is defined as the instantaneous rate of change of the function at a point. The derivative at the point x_0 is:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

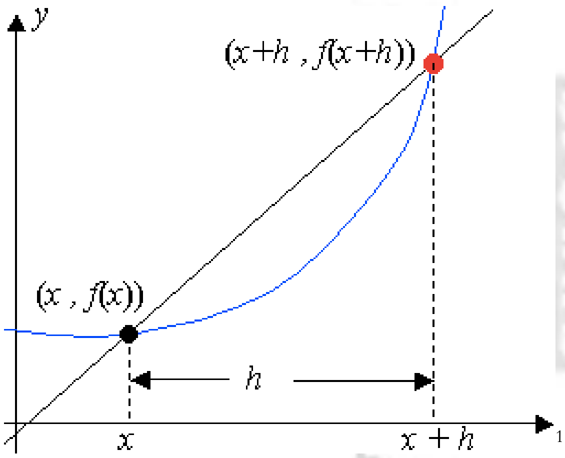
where h is the change between the initial value of x_0 and final value of $x_0 + h$. The definition of the derivative may be thought of as an extension of the slope formula (or "average rate of change" formula):

$$\frac{\Delta y}{\Delta x} = \frac{f(x)}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

If we designate $x_1 = x_0 + h$ where h represents the change between x_1 and x_0 then:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0} = \frac{f(x_0 + h) - f(x_0)}{h}$$

Graphically, this equation is the line (known as the "secant line") that connects two points of the function, as seen in the figure at the top of the page:



[1]

The limit as h approaches 0 returns the derivative at $(x_0, f(x_0))$:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

The derivative of the function is defined as instantaneous rate of change of the function at a point. Possible notations include:

$$\frac{dy}{dx} \text{ or } \frac{d(f(x)}{dx} \text{ or } \frac{d}{dx} f(x) \text{ or } f'(x)$$

$\frac{d}{dx}$ represents the process of differentiation[2] of the function $f(x)$. $f'(x)$ is another way to represent differentiation and is referred to as prime notation.

Information on how to calculate the derivative using the definition may be found here:
http://carp.sfsu.edu/sites/sites7.sfsu.edu.carp/files/The Limit Definition of the Derivative.pdf

In practice, the definition is rarely used to find the derivative. Instead, we use various rules ("derivative rules") to find the derivative more efficiently. The remainder of this handout is dedicated to introducing some of these derivative rules.

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[1] Image source: http://www.teacherschoice.com.au/first_principles.htm

[2] Differentiation is the operation of taking the derivative. We say that the function is "differentiable" at a point if "it has the derivative at a point". Accordingly, "to differentiate a function" is "to find a derivative of a function". "Derivative" corresponds to verb "differentiate", not "derive".

Differentiation rules

This section will:

- Introduce the following differentiation rules:
    - Derivative of a constant function
    - Power rule
    - Constant multiple derivative rule
    - Derivative sum rule
    - Product rule
    - Quotient rule
    - Derivative of natural exponential function
- Provide examples of application of these rules for finding derivatives
- Provide examples where some of these rules must be combined in order to find the derivative of a function.

Differentiation rules are efficient methods of finding the derivative without using the definition of derivative. Depending on the kind of function we are given, specific differentiation rules must be applied in order to find the derivative.

Derivative of a constant function:

Let c be a constant. The derivative of c is:

$$\frac{d}{dx}c = 0$$

To understand this rule intuitively, recall that derivatives measure instantaneous rate of change of a function at a point. The output of constant functions does not change, and so their instantaneous rate of change is always zero. For example, suppose we are given the function

$$f(x) = 5$$

then the derivative of that function with respect to x is

$$\frac{d}{dx}f(x) = 0$$

Since the function is constant, the rate of change is 0, and so the instantaneous rate of change (i.e. the derivative) is also 0.

Suppose we are given the function

$$f(x) = \sqrt[3]{\left(3 - \frac{1}{2}\right)^2} \approx 1.842$$

$\sqrt[3]{\left(3 - \frac{1}{2}\right)^2}$ is a constant function as well, and so its derivative is 0:

$$\frac{d}{dx}\sqrt[3]{\left(3-\frac{1}{2}\right)^2} = 0$$

Power rule

Power rule assists us in finding the derivative of functions where x is raised to some constant power n. The general form of Power rule is the following:

$$\frac{d}{dx}x^n = nx^{n-1}$$

For example, suppose we are asked to find the derivative of the function below:

$$f(x) = x^3$$

The function $f(x) = x^3$ is of the form x^n, which means that we may find the derivative of using Power rule. Specifically:

$$\frac{d}{dx}(x^3) = 3x^{3-1}$$

$$= 3x^2$$

As another example, suppose we are asked to find the derivative of the function given below:

$$f(x) = \sqrt{x}$$

We may rewrite the function in the form of x^n where n is a fractional power[3]:

$$f(x) = x^{\frac{1}{2}}$$

We may use the power rule to find the derivative:

$$\frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2}x^{\left(\frac{1}{2}-1\right)}$$

$$= \frac{1}{2}x^{-\frac{1}{2}}$$

By using laws of exponents, namely $x^{-n} = \frac{1}{x^n}\ \sqrt[n]{x} = x^{\frac{1}{n}}$, the previous expression may also be written as

$$\frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

[3] All expressions of the kind $\sqrt[n]{(x)^c}$ can be represented as $x^{\frac{c}{n}}$ (note $\sqrt{x} \equiv \sqrt[2]{x}$) For example, $\sqrt[2]{x^1}$, which is generally represented as "the square root of x" or $\sqrt{x}$, is equal to $x^{\frac{1}{2}}$. Similarly, $\sqrt[3]{x^2} = x^{\frac{2}{3}}$.

Suppose we are given the following function:

$$f(x) = x^{\frac{\sqrt{\pi - \frac{3}{4}}}{7}}$$

Even though x raised to the power which is represented not as an integer, $\frac{\sqrt{\pi-\frac{3}{4}}}{7}$ is still some constant number, since it consists of only constant values. Therefore, we can apply the power rule here as well:

$$\frac{d}{dx}f(x) = \left(\frac{\sqrt{\pi - \frac{3}{4}}}{7}\right)x^{\left(\frac{\sqrt{\pi-\frac{3}{4}}}{7}-1\right)}$$

Constant multiple rule

Constant multiple rule tells us that if our function u contains some constant multiple c, then c can be taken out of the derivative:

$$\frac{d}{dx}(cu) = c\frac{d}{dx}(u)$$

For example, we are asked to find the derivative of the following function:

$$f(x) = 3x^2$$

To find the derivative, we will apply the constant multiple rule and power rule:

$$\frac{d}{dx}(3x^2) = 3\frac{d}{dx}(x^2)$$

$$= 3(2)x^{2-1}$$

$$= 6x$$

As one last example, suppose we are asked to find the derivative of the following function:

$$f(x) = \left(\frac{\pi - 2}{n}\right)x^n$$

where n and π are constant values. $\left(\frac{\pi-2}{n}\right)$ is some constant value since it contains only constants. We apply the constant multiple rule to take $\left(\frac{\pi-2}{n}\right)$ out of derivative, and then find the derivative of x^n using power rule:

$$\frac{d}{dx}\left(\left(\frac{\pi-2}{n}\right)x^n\right)=\left(\frac{\pi-2}{n}\right)\frac{d}{dx}(x^n)$$

$$=\left(\frac{\pi-2}{n}\right)nx^{n-1}$$

$$=(\pi-2)x^{n-1}$$

Derivative Sum Rule and Difference Rule

Derivative sum rule says that the derivative of $u+v$ will the derivative of u plus the derivative of v, symbolically[4]:

$$\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}$$

For example, suppose we to find the derivative of the following function:

$$f(x)=x^3+34$$

we can think of this function as the combination of the two following functions: $u(x)=x^3$ and $v(x)=34$

$$f(x)=u(x)+v(x)$$

$$=x^3+34$$

We may use the derivative sum rule to rearrange out derivative as follows

$$\frac{d}{dx}(x^3+34)=\frac{d}{dx}(x^3)+\frac{d}{dx}(34)$$

Now we can use power rule and derivative of a constant function rule to find the derivative of $\frac{d}{dx}x^3$and $\frac{d}{dx}34$

$$\frac{d}{dx}(x^3+34)=\frac{d}{dx}(x^3)+\frac{d}{dx}(34)$$

$$=3x^2+0$$

$$=3x^2$$

[4] An additional condition is u and v are differentiable (i.e. their derivative exists). We will assume all functions are differentiable unless otherwise stated.

The difference rule is obtained from derivative sum rule:

$$\frac{d}{dx}(u-v) = \frac{d}{dx}\big(u+(-v)\big)$$

$$= \frac{du}{dx} + \frac{d(-v)}{dx}$$

$$= \frac{du}{dx} - \frac{dv}{dx}$$

Derivative of the Natural exponential function

The derivative of the natural exponential function (the function $f(x) = e^x$) formula helps us find the derivative of e^x without using the definition of the derivative. e is a constant; the value of e up to 2 decimal places is: $e \approx 2.72$ similar to how $\pi \approx 3.14$. The derivative of e^x is:

$$\frac{d}{dx}(e^x) = e^x$$

That is, e^x is its own derivative.

Suppose we are given the function:

$$f(x) = 2e^x$$

We will use constant multiple rule and derivative of e^x rule to find the derivative of $f(x)$

$$\frac{d}{dx}(2e^x) = 2\frac{d}{dx}e^x$$

$$= 2e^x$$

Product rule

The product rule says the derivative of uv is u times the derivative of v plus v times the derivative of u. Symbolically:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

We use product rule whenever we have the product of two functions that are differentiable with respect to x.

For example, suppose we want to find the derivative of the function

$$f(x) = (x^2)(3x+1).$$

The function is a product of two functions: $u(x) = x^2$ and $v(x) = 3x+1$. Applying the product rule:

$$\frac{d}{dx}[(x^2)(3x+1)] = x^2\left[\frac{d}{dx}(3x+1)\right] + \left[\frac{d}{dx}(x^2)\right](3x+1)$$

next, we apply power rule and sum rule to solve for derivative of $v(x) = 3x + 1$ and $u(x) = x^2$:

$$x^2\left[\frac{d}{dx}(3x+1)\right] + \left[\frac{d}{dx}(x^2)\right](3x+1) = 3x^2 + 2x(3x+1)$$

$$= 3x^2 + 6x^2 + 2x$$

$$= 9x^2 + 2x$$

We may verify if our result is correct by expanding our function $f(x) = (x^2)(3x+1)$ and solving for derivative using the sum rule and power rule:

$$f(x) = (x^2)(3x+1)$$

$$= 3x^3 + x^2$$

$$\frac{d}{dx}(3x^3 + x^2) = 9x^2 + 2x$$

Both results agree, as expected.

Not all functions can be simplified to avoid using the power rule. For example:

$$f(x) = 2xe^x$$

We must use the power rule to differentiate this function since we the function cannot be simplified.

$$u(x) = 2x \quad v(x) = e^x$$

$$\frac{d}{dx}(2xe^x) = 2x\left[\frac{d}{dx}(e^x)\right] + \left[\frac{d}{dx}(2x)\right]e^x$$

To solve for derivative we use the derivative of the natural exponential function rule and power rule to find derivatives of $v(x) = e^x$ and $u(x) = 2x$

$$2x\left[\frac{d}{dx}(e^x)\right] + \left[\frac{d}{dx}(2x)\right]e^x = 2xe^x + 2e^x$$

Quotient rule

Suppose we have a function $f(x)$ that can be decomposed into the quotient of two simpler functions in the form of $f(x) = \frac{u(x)}{v(x)}$. To find the derivative of $f(x)$ we use the following equation

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{u(x)}{v(x)}\right)$$

$$= \frac{\left[\frac{d}{dx}u(x)\right]v(x) - u(x)\left[\frac{d}{dx}v(x)\right]}{[v(x)]^2}$$

or in prime notation

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

The quotient rule is useful for finding the derivative of functions that are presented in the fractional form. Consider the following function

$$f(x) = \frac{x^2 - 2}{x^3}$$

Since this function is given in the quotient form we can apply the quotient rule to find the derivative of the function. We can think of the numerator as our $u(x) = x^2 - 2$ and the denominator as our $v(x) = x^3$. Applying the quotient rule:

$$\frac{d}{dx}\left(\frac{x^2 - 2}{x^3}\right) = \frac{\left[\frac{d}{dx}(x^2 - 2)\right](x^3) - (x^2 - 2)\left[\frac{d}{dx}(x^3)\right]}{(x^3)^2}$$

We apply the power rule and the sum rule to find the derivatives:

$$\frac{\left[\frac{d}{dx}(x^2 - 2)\right](x^3) - (x^2 - 2)\left[\frac{d}{dx}(x^3)\right]}{(x^3)^2} = \frac{(2x)(x^3) - (x^2 - 2)(3x^2)}{x^6}$$

$$= \frac{6x^2 - x^4}{x^6}$$

$$= \frac{6 - x^2}{x^4}$$

Conclusion:

In this handout we have done the following:

- Reviewed the definition and notation of the derivative
- Explored various differentiation rules
- Provided some examples of application of these rules, and also examples where we had to use several rules to find the derivative. See following page for practice problems.
- Provided a list of derivative rules (see last page of document).

Practice problem: Find derivatives of the following functions (answers on the next page):

1. $f(x) = 5x^{\frac{3}{5}} - 47$

2. $f(x) = (x+5)(x^2 - 4x)$

3. $f(x) = 3xe^x + \frac{\sqrt{x^3}}{e^x}$

Answers to Practice problems:

1) $f'(x) = 3x^{-\frac{2}{5}}$

2) $f'(x) = 3x^2 + 2x - 20$

3) $f'(x) = 3e^x + 3xe^x - e^{-x}\sqrt{x^3} + \frac{3}{2}e^{-x}\sqrt{x}$

Reference:
George B. Thomas, Jr "Thomas' Calculus Early Transcendentals" Twelfth Edition, 2010

Summary of Differentiation rules:

Derivative of Constant Function	$\frac{d}{dx}(c) = 0$ (Where c is a constant number)
Power Rule	$\frac{d}{dx}(x^n) = nx^{n-1}$ (Where n is a constant number)
Constant Multiple Rule	$\frac{d}{dx}(cx) = c\frac{d}{dx}(x)$ (where c is a constant number)
Derivative Sum Rule/Difference Rule	$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$ $\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$
Derivative of Natural Exponential Function	$\frac{d}{dx}(e^x) = e^x$
Product Rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
Quotient Rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$