Campus Academic Resource Program

The Limit Definition of the Derivative

This Handout will:

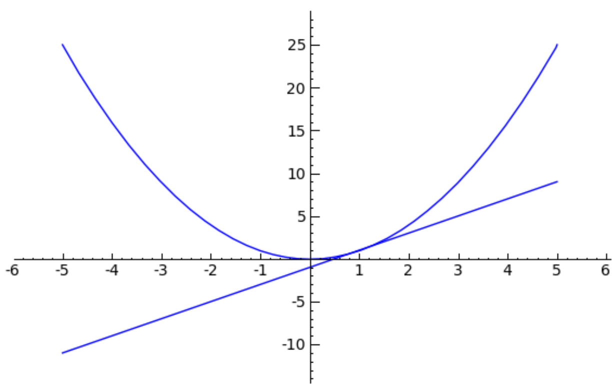
- Define the limit graphically and algebraically
- Discuss, in detail, specific features of the definition of the derivative
- Provide a general strategy of finding the derivative by definition
- Demonstrate the process of finding the derivative by definition using four distinct examples.

<u>The Derivative</u>

This section will:

- Introduce the graphical limit of a derivative
- Define the derivative algebraically as a limit
- Provide strategies for handling specific features of the algebraic definition

The derivative is the primary topic of calculus I. The derivative is the *instantaneous rate of change* of a function at any point. Graphically, the derivative is the slope of the tangent line through the point. The tangent line is a line that passes through two infinitesimally close points on a curve. This can be thought of as "just touching" the curve at a point. This is pictured below.



[1]

More information on the graphical definition of the derivative can be found in "§3.1 Tangents and the Derivative at a Point" in Thomas Calculus.

Algebraically, the derivative can be found by taking a particular limit, called the **limit definition of the derivative**:

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$	Definition of the Derivative

Where $f'(x)$ is the derivative, $f(x)$ is the function we wish to **differentiate**[2], and h is a dummy variable that will not appear in the final result.

[1] Image source: http://www.norsemathology.org/wiki/images/9/9f/Sage_tangent_line.png

[2] "Differentiate" means "to find the derivative of." Differentiation is the process of finding a derivative.

That is, we can find any derivative by substituting the function into the "Definition of the Derivative" and evaluating the limit as h approaches zero. Practically, to evaluate the limit we will first evaluate $f(x+h)$. For example, if

$$f(x) = x^2 + 4.$$

We may evaluate $f(x+h)$ if we treat $(x+h)$ as a quantity that may be substituted into the function—that is, wherever we replace every x with $(x+h)$. For this function, $f(x+h)$ would be:

$$f(x+h) = (x+h)^2 + 4$$

See "Determining $f(x+h)$" for more examples on this process.

After we evaluate $f(x+h)$, we then **manipulate** the resulting expression until we may substitute in $h = 0$ without obtaining a $\frac{0}{0}$ form. For example, if

$$f'(x) = \lim_{h \to 0} \left(\frac{x^2 + 2xh - x^2}{h} + h \right)$$

Substituting in $h = 0$ evaluates to $\frac{x^2+0-x^2}{0} + 0 = \frac{0}{0}$, so we need to manipulate ("simplify") the expression algebraically. Simplifying

$$\frac{x^2 + 2xh - x^2}{h} + h = \frac{2xh}{h} + h = 2x + h$$

The limit

$$f'(x) = \lim_{h \to 0} (2x + h)$$

may be evaluated by directly substituting in $h = 0$ to obtain $2x$. Do "Problem 1" on practice problems for more practice on this process.

Summary: When evaluating the derivative:

- First evaluate $f(x+h)$ so that it may be substituted into the definition of the derivative
- Manipulate the resulting expression (i.e. from the definition of the derivative) until you may substitute $h = 0$ without obtaining a $\frac{0}{0}$ form.

<u>Determining $f(x+h)$</u>

This section will provide three more examples on evaluating expressions of the form $f(x+h)$ for these following functions:

$$f_1(x) = x^3 - 2x$$

$$f_2(x) = \sqrt{x+3}$$

$$f_3(x) = \frac{2}{x^2}$$

To evaluate, we treat $(x+h)$ as a quantity and place $(x+h)$ directly in the function in place of x, as follows:

$$f_1(x+h) = (x+h)^3 - 2(x+h)$$

$$f_2(x+h) = \sqrt{(x+h)+3}$$

$$f_3(x+h) = \frac{2}{(x+h)^2}$$

Finding the Derivative:

This section will

- Provide a general strategy for finding derivatives by definition.
- Provide specific strategies for four distinct worked examples.

Examples are the best way to learn how to take derivatives by definition. The following four examples will illustrate this process of finding the derivative. The general process of each example is:

- Start with the definition of the derivative
- Explicitly write out $f(x+h)$ and $f(x)$
- Manipulate the expression algebraically ("simplify"). *This is the step that changes the most from example-to-example and each example will have its own algebraic manipulate.*
- Continue to manipulate the algebra until we may substitute in $h = 0$ without obtaining a $\frac{0}{0}$ form.
- Substitute in $h = 0$.
- Simplify the result as much as possible.

We will find the derivatives of these following functions:

These examples are chosen to provide as much diversity as possible. Example 1 is chosen to illustrate the method, examples 2-4 are chosen to illustrate specific methods of algebraic manipulation.

Example 1: We will find the derivative of $f(x) = x^2 + 4$. Write the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Substitute in our functions for $f(x+h)$ and $f(x)$:

$$f'(x) = \lim_{h \to 0} \frac{((x+h)^2 + 4) - (x^2 + 4)}{h}$$

Expand the factors in the numerator:

$$f'(x) = \lim_{h \to 0} \frac{x^2 + 2hx + h^2 + 4 - x^2 - 4}{h}$$

Combine like terms in the numerator.

$$f'(x) = \lim_{h \to 0} \frac{2hx + h^2}{h}$$

Factor out the h in the numerator.

$$f'(x) = \lim_{h \to 0} \frac{h(2x + h)}{h}$$

Cancel out the h in the numerator and denominator.

$$f'(x) = \lim_{h \to 0} 2x + h$$

Substitute in $h = 0$, as we may substitute $h = 0$ without obtaining a $\frac{0}{0}$ form

$$f'(x) = 2x + 0$$

This gives us our final result: the derivative:

$$f'(x) = 2x$$

Example 2: Simple Rational Function[3]

Let

$$f(x) = \frac{1}{x+3}$$

To find the derivative, substitute $f(x)$ into the definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h}$$

Reduce the number of fractions into the expression

$$\frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h} = \frac{1}{h}\left(\frac{1}{x+h+3} - \frac{1}{x+3}\right)$$

by factoring out $\frac{1}{h}$. This is similar to how we can write $\frac{3}{2} = \frac{1}{2} \cdot (3)$ or $\frac{x}{y} = \frac{1}{y} \cdot x$, except our numerator in this case has fractions. Our derivative is

$$f'(x) = \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{x+h+3} - \frac{1}{x+3}\right).$$

Simplify $\frac{1}{x+h+3} - \frac{1}{x+3}$. To do this, we want both fractions to have a common denominator and follow standard fraction arithmetic where $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$. If you are unclear on how to add fractions, refer to the CARP handout on fractions[4]. Simplifying

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{x+h+3}\left(\frac{x+3}{x+3}\right) - \frac{1}{x+3}\left(\frac{x+h+3}{x+h+3}\right)\right) \\
&= \lim_{h \to 0} \frac{1}{h}\left(\frac{x+3}{(x+h+3)(x+3)} - \frac{x+h+3}{(x+h+3)(x+3)}\right) \\
&= \lim_{h \to 0} \frac{1}{h}\left(\frac{x+3-(x+h+3)}{(x+h+3)(x+3)}\right) \\
&= \lim_{h \to 0} \frac{1}{h} \frac{-h}{(x+h+3)(x+3)} \\
&= \lim_{h \to 0} \frac{-1}{(x+h+3)(x+3)}
\end{aligned}$$

In the last step, we may substitute in $h = 0$ without leading to a $\frac{0}{0}$ form, and in doing so we obtain

$$f'(x) = \frac{-1}{(x+3)^2}$$

[3] A rational function is defined as a quotient of two polynomial functions. This rational function is "simple" because the numerator is 1, which is the simplest polynomial.
[4] https://sites7.sfsu.edu/sites/sites7.sfsu.edu.carp/files/PDF/Math/Algebral-Math60/IntroductiontoFractions.pdf

Example 3: Radical Expression

$$f(x) = \sqrt{2x+5}$$

To find the derivative, substitute $f(x)$ into the definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{2(x+h)+5} - \sqrt{2x+5}}{h}$$

We want to rewrite the above expression without square roots which requires the concept of a **conjugate**. A conjugate of an expression means that you take that expression and change a sign. For $\sqrt{q} - \sqrt{p}$, the conjugate is $\sqrt{q} + \sqrt{p}$. The product $\left(\sqrt{q} - \sqrt{p},\right)\left(\sqrt{q} + \sqrt{p}\right) = p + q$ by **difference of squares**, which you may verify by using FOIL on the expression $\left(\sqrt{q} - \sqrt{p},\right)\left(\sqrt{q} + \sqrt{p}\right)$. We will refer to this process as **conjugation.**

The conjugate of $\sqrt{2(x+h)+5} - \sqrt{2x+5}$ is $\sqrt{2(x+h)+5} + \sqrt{2x+5}$. To multiply the numerator by $\sqrt{2(x+h)+5} + \sqrt{2x+5}$, also multiply the denominator by $\sqrt{2(x+h)+5} + \sqrt{2x+5}$:

$$f'(x) = \lim_{h \to 0} \left(\frac{\sqrt{2(x+h)+5} - \sqrt{2x+5}}{h}\right)\left(\frac{\sqrt{2(x+h)+5} + \sqrt{2x+5}}{\sqrt{2(x+h)+5} + \sqrt{2x+5}}\right)$$

The numerator is $2(x+h) + 5 - (2x+5)$ by difference of squares.

$$f'(x) = \lim_{h \to 0} \frac{2(x+h)+5-(2x+5)}{h\left(\sqrt{2(x+h)+5} + \sqrt{2x+5}\right)}$$

$$= \lim_{h \to 0} \frac{2x+2h+5-2x-5}{h\left(\sqrt{2(x+h)+5} + \sqrt{2x+5}\right)}$$

$$= \lim_{h \to 0} \frac{2h}{h\left(\sqrt{2(x+h)+5} + \sqrt{2x+5}\right)}$$

$$= \lim_{h \to 0} \frac{2}{\left(\sqrt{2(x+h)+5} + \sqrt{2x+5}\right)}$$

Substitute in $h = 0$, as you may do so without obtaining a $\frac{0}{0}$ indeterminate form:

$$f'(x) = \frac{2}{\left(\sqrt{2x+5} + \sqrt{2x+5}\right)}$$

$$= \frac{2}{2\sqrt{2x+5}}$$

$$= \frac{1}{\sqrt{2x+5}}$$

Example 4: Mixing fractions and radicals.
Let

$$f(x) = \frac{1}{\sqrt{1-x}}$$

Techniques present in both example 2 and example 3 must be used and should be reviewed and understood before proceeding with this example. Substitute $f(x)$ into the definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{\sqrt{1-(x+h)}} - \frac{1}{\sqrt{1-x}}}{h}$$

Using the same technique in example 2, we may rewrite the previous expression as

$$f'(x) = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{\sqrt{1-(x+h)}} - \frac{1}{\sqrt{1-x}}\right)$$

Use conjugation to simplify the square roots. The conjugate of $\frac{1}{\sqrt{1-(x+h)}} - \frac{1}{\sqrt{1-x}}$ is $\frac{1}{\sqrt{1-(x+h)}} + \frac{1}{\sqrt{1-x}}$.
Multiply:

$$f'(x) = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{\sqrt{1-(x+h)}} - \frac{1}{\sqrt{1-x}}\right)\frac{\left(\frac{1}{\sqrt{1-(x+h)}} + \frac{1}{\sqrt{1-x}}\right)}{\left(\frac{1}{\sqrt{1-(x+h)}} + \frac{1}{\sqrt{1-x}}\right)}$$

Simplify the product $\left(\frac{1}{\sqrt{1-(x+h)}} - \frac{1}{\sqrt{1-x}}\right)\left(\frac{1}{\sqrt{1-(x+h)}} + \frac{1}{\sqrt{1-x}}\right)$ using difference of squares:

$$f'(x) = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{1-(x+h)} - \frac{1}{1-x}\right)\frac{1}{\left(\frac{1}{\sqrt{1-(x+h)}} + \frac{1}{\sqrt{1-x}}\right)}$$

We will not be able to simplify $\frac{1}{\left(\frac{1}{\sqrt{1-(x+h)}} + \frac{1}{\sqrt{1-x}}\right)}$ until the last step. Simplify $\left(\frac{1}{1-(x+h)} - \frac{1}{1-x}\right)$ by using standard fraction arithmetic like that used in example 2:

$$f'(x) = \lim_{h\to 0} \frac{1}{h}\left(\frac{1-x}{\big(1-(x+h)\big)(1-x)} - \frac{1-(x+h)}{\big(1-(x+h)\big)(1-x)}\right)\frac{1}{\left(\frac{1}{\sqrt{1-(x+h)}} + \frac{1}{\sqrt{1-x}}\right)}$$

$$f'(x) = \lim_{h\to 0} \frac{1}{h}\left(\frac{1-x-\big(1-(x+h)\big)}{\big(1-(x+h)\big)(1-x)}\right)\frac{1}{\left(\frac{1}{\sqrt{1-(x+h)}} + \frac{1}{\sqrt{1-x}}\right)}$$

Simplify as much as possible:

$$f'(x) = \lim_{h\to 0} \frac{1}{h}\left(\frac{1-x-1+(x+h)}{\big(1-(x+h)\big)(1-x)}\right)\frac{1}{\left(\frac{1}{\sqrt{1-(x+h)}} + \frac{1}{\sqrt{1-x}}\right)}$$

$$f'(x) = \lim_{h\to 0} \frac{1}{h}\left(\frac{1-x-1+x+h}{\big(1-(x+h)\big)(1-x)}\right)\frac{1}{\left(\frac{1}{\sqrt{1-(x+h)}}+\frac{1}{\sqrt{1-x}}\right)}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{h}{\big(1-(x+h)\big)(1-x)}\right)\frac{1}{\left(\frac{1}{\sqrt{1-(x+h)}}+\frac{1}{\sqrt{1-x}}\right)}$$

$$= \lim_{h\to 0} \left(\frac{1}{\big(1-(x+h)\big)(1-x)}\right)\frac{1}{\left(\frac{1}{\sqrt{1-(x+h)}}+\frac{1}{\sqrt{1-x}}\right)}$$

Substitute in $h = 0$, as we may do so without obtaining a $\frac{0}{0}$ indeterminate formL

$$f'(x) = \frac{1}{(1-x)(1-x)}\frac{1}{\left(\frac{1}{\sqrt{1-x}}+\frac{1}{\sqrt{1-x}}\right)}$$

$$= \frac{1}{(1-x)(1-x)}\frac{1}{\left(\frac{2}{\sqrt{1-x}}\right)}$$

We may now simplify $\frac{1}{\left(\frac{2}{\sqrt{1-x}}\right)}$;

$$f'(x) = \frac{1}{(1-x)(1-x)}\frac{1}{\left(\frac{2}{\sqrt{1-x}}\right)}\frac{\frac{\sqrt{1-x}}{2}}{\frac{\sqrt{1-x}}{2}}$$

$$= \frac{1}{(1-x)(1-x)}\frac{\sqrt{1-x}}{2}$$

$$= \frac{\sqrt{1-x}}{2(1-x)^2}$$

We may simplify the previous expression even further by noting $\sqrt{1-x} = (1-x)^{\frac{1}{2}}$, so the previous expression becomes

$$f'(x) = \frac{(1-x)^{\frac{1}{2}}}{2(1-x)^2}$$

We may apply laws of exponents[5]

$$f'(x) = \frac{1}{2}(1-x)^{-\frac{3}{2}} = \frac{1}{2(1-x)^{\frac{3}{2}}},$$

which is a form you obtain if you were to apply derivative rules.

[5] Refer to the formula sheet in front cover of calculus textbook for Laws of Exponents

Glossary:

This section will define key terms used in this section

Algebraic Manipulation: Often called simplifying. Refers to the process of rewriting an algebraic expression in an equivalent expression that is more manageable. E.g., $\frac{x^2+x}{x} = x + 1$ is an example of algebraic manipulation.

Conjugate: A quantity that is similar to another quantity except for a sign change. For this handout, we deal with expressions $\sqrt{q} + \sqrt{p}$, which has a conjugate $\sqrt{q} - \sqrt{p}$.

Conjugation: The process of simplifying an expression by multiplying it by its conjugate.

Definition of Derivative: Also known as algebraic or limit definition of the derivative. Refers to the equation $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$.

Difference of Squares: The identity that says $(x^2 - y^2) = (x + y)(x - y)$. A particularly useful instance of this identity for this handout is $x - y = (\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})$.

Practice Problems:

Problem 1: Which of the following limits may be evaluated by substituting in $h = 0$ without obtaining a $\frac{0}{0}$ form? (Answers given at bottom of page)

1.) $\lim_{h \to 0} \frac{(x+h)^2-(x-h)^2}{2h}$

2.) $\lim_{h \to 0} \frac{x+h}{x^2-h}$

3.) $\lim_{h \to 0} \frac{h}{h}$

Problem 2: For each of the following functions, write out $f(x + h)$

1.) $f(x) = x^2 + 1$
2.) $f(x) = \sqrt{1 - x}$
3.) $f(x) = \frac{1}{2x+4}$
4.) $f(x) = \frac{1}{\sqrt{x}}$

Answers for problem 1: 1.) No, 2.) Yes, 3.) No

For problems 3 to 6, find the derivative of the following problems.

Problem 3: Find the derivate of $f(x) = x^2 + 1$

Problem 4: Find the derivative of $f(x) = \sqrt{1 - x}$

Problem 5: Find the derivative of $f(x) = \frac{1}{2x+4}$

Problem 6: Find the derivative of $f(x) = \frac{1}{\sqrt{x}}$

Citations:

Thomas, G. B., Weir, M. D., Hass, J., & Giordano, F. R. (2010). *Thomas' Calculus Early Transcendentals*. Pearson.