This handout will:

- Discuss the concept of function composition.
- Provide examples of combining multiple functions.
- Provide an example how to find the domain of a function composition.

The reader should be familiar with domain and range of functions.

**Motivation for Understanding Composition:**
This section will explain why composition functions are important.

Precalculus is the study of functions and their behavior. To understand a wide variety of functions, we study both unit functions¹, whose properties must be memorized, and combinations, which include sums, differences, products, quotients and compositions of unit functions. Combinations are important in calculus, as important rules (“derivative rules”) are defined specifically only for unit functions and generically for combinations. Addition, subtraction, products, and quotients correspond to the classical operations for numbers (e.g. the sum of $f(x) = x^2$ and $g(x) = x^3$ is $(f + g)(x) = x^2 + x^3$). Composition, on the other hand, is an operation uniquely defined for functions.

Knowing how to dissect a combination into its unit functions is an equally important skill since calculus rules are defined generically for combinations. While most examples will show to compose two functions to illustrate what composition is as an operator, a more important long-term skill is to examine a composition function and decompose it into its original functions. Refer to section 1.4 “Decomposing Functions” in Axler’s *Precalculus: A Prelude to Calculus*, as this handout does not discuss function decomposition.

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¹ See Appendix 1 for definition of unit functions. Note that the term unit functions is defined only for the purposes of this handout and the definition is unlikely to be found in any precalculus textbook.
Definition and Notation of Composite Functions:

This section will:

- Define Function Composition
- Show Notations for Composition

**Definition:** A composite function is a function whose input is another function. A composition is computed by replacing the $x$ variable of one function with the new input function $g(x)$. For example, if $f(x) = \sqrt{x}$ and $g(x) = 1 + x^2$, then $f(g(x)) = \sqrt{1 + x^2}$ is a composition of $\sqrt{x}$ and $1 + x^2$—the input of $f(x)$ is another function $g(x)$. Examples in the next section show types of composite functions you may encounter. The term composition refers to the act of composing two functions to make a composite function.

**Notation:** There are two canonical notations used for composition. Let $h(x)$ be a composition of $f(x)$ and $g(x)$, then we may write

\[ h(x) = f(g(x)) \]

or

\[ h(x) = (f \circ g)(x). \]

In either case, we pronounce the composition as “$f$” of “$g$”. Each notation has different emphases:

The $(f \circ g)(x)$ notation emphasizes the combination nature of compositions, e.g. by following the same form as sums/differences, $(f \pm g)(x); f(g(x))$ emphasizes the mathematical procedure of substituting $g(x)$ in place of $x$ in the function $f(x)$. We will primarily use the $f(g(x))$ notation throughout this handout.
Worked Examples—Composing Functions:

This section will provide worked examples of composing two functions.

Example 1

Let \( f(x) = \frac{1}{x} + 2 \) and \( g(x) = x^2 \), find \( f(g(x)) \).

\[
\begin{align*}
f(g(x)) &= f(g(x)) \\ &= \frac{1}{g(x)} + 2 \quad \text{Substitute } g(x) \text{ for } x \text{ in } f(x) \\ &= \frac{1}{x^2} + 2 \quad \text{Substitute in } g(x) \\ &= \frac{1}{x^2} + 2 \quad \text{Simplify}
\end{align*}
\]

Conclusion: \( f(g(x)) = \frac{1}{x^2} + 2 \). The above method outlines a typical calculation used to compute composition functions.

Example 2

Let \( f(x) \) and \( g(x) \) be the same as in example 1, but instead of finding \( f(g(x)) \), find \( g(f(x)) \).

\[
\begin{align*}
g(f(x)) &= g(f(x)) \\ &= (f(x))^2 \quad \text{Substitute } f(x) \text{ for } x \text{ in } g(x) \\ &= \left(\frac{1}{x} + 2\right)^2 \quad \text{Substitute in } f(x) \\ &= \frac{1}{x^2} + \frac{4}{x} + 4 \quad \text{Expand} \\ &= \frac{1}{x^2} + \frac{4}{x} + 4 \quad \text{Simplify}
\end{align*}
\]

Conclusion: \( g(f(x)) = \frac{1}{x^2} + \frac{4}{x} + 4 \) in this example and \( f(g(x)) = \frac{1}{x^2} + 2 \): \( f(g(x)) \neq g(f(x)) \), similar to how \( 4 - 3 \neq 3 - 4 \) and \( \frac{2}{3} \neq \frac{3}{2} \).

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\(^2\) The phrase “substitute \( g(x) \) for \( x \) in \( f(x) \)” means replace the \( x \) found in expression \( f(x) \) with \( g(x) \). Future examples will use this phrase frequently.
Example 3

Let \( f(x) = x^2 + x \) and \( g(x) = 1 + x \), find \( f(g(x)) \)

\[
\begin{align*}
  f(g(x)) &= f(g(x)) \\
  &= (g(x))^2 + g(x) & \text{Substitute } g(x) \text{ for } x \text{ in } f(x) \\
  &= (1 + x)^2 + (1 + x) & \text{Substitute in } g(x) \\
  &= 1 + 2x + x^2 + 1 + x & \text{Expand} \\
  &= x^2 + 3x + 2 & \text{Simplify}
\end{align*}
\]

Conclusion: \( f(g(x)) = x^2 + 3x + 2 \).

Composing of More Than Two Functions:

This section will:

- Define composition of more than two functions
- Provide an example thereof.

We sometimes are faced with the problem of finding the composition more than two functions, \( f \circ g \circ h \equiv f \left( g(h(x)) \right) \).

To find the composition, we first substitute \( g(h(x)) \) for \( x \) in \( f(x) \), and then substitute \( h(x) \) for \( x \) in \( g(x) \), and then substitute in \( h(x) \). Example 4 on the next page provides an example of this process.
Example 4

Let \( f(x) = x - 10 \), \( g(x) = x^2 \), and \( h(x) = x - 2 \), find \( f(g(h(x))) \).

\[
f(g(h(x))) = f(g(h(x)))
= g(h(x)) - 10 \quad \text{Substitute } g(h(x)) \text{ for } x \text{ in } f(x)
= (h(x))^2 - 10 \quad \text{Substitute } h(x) \text{ for } x \text{ in } g(x)
= (x - 2)^2 - 10 \quad \text{Substitute equation in for } h(x)
= x^2 - 4x + 4 - 10 \quad \text{Expand equation}
= x^2 - 4x - 6 \quad \text{Simplify equation}
\]

Conclusion: \( f(g(h(x))) = x^2 - 4x - 6 \).

Domains of Composite Functions:

This section will:

- Outline a three-step process to find domain of functions
- Provide an example using this three-step process.

Formally, the domain of \( f(g(x)) \) is the set of real numbers that are in the domain of \( g(x) \) and satisfy the domain of \( f(g(x)) \). A three-step process for finding the domain is:

Step 1. First identify what value(s) of \( g(x) \) are not in the domain of \( f(x) \).

Step 2. Find what value(s) of \( x \) do not produce the values of \( g(x) \) described in step 1.

Step 3. The domain of \( f(g(x)) \) is the domain of \( g(x) \) excluding the \( x \) values found in the previous step and excluding \( x \) values not in \( g(x) \)'s domain.
Example 5

Let \( f(x) = \frac{1}{x-13} \) with domain \( 1 < x \leq 8 \) and \( g(x) = x^2 + 4 \) with domain \( 0 \leq x \leq 4 \). Find the composition and domain of \( f(g(x)) \)

\[
f(g(x)) = \frac{1}{g(x)-13}
\]

Substitute \( g(x) \) for \( x \) in \( f(x) \)

\[
= \frac{1}{(x+4)-13}
\]

Substitute in \( g(x) \)

\[
= \frac{1}{x^2+4-5}
\]

Clear parentheses

\[
= \frac{1}{x^2-9}
\]

Simplify denominator

Although we found the expression \( f(g(x)) = \frac{1}{x^2-9} \), the problem is not complete until we have determined the domain. We will use the three-step process outlined on the previous page.

Step 1: The first step is to identify what values of \( g(x) \) are not in the domain of \( f(x) \). Since \( f(x) \) has a domain \( 1 < x \leq 8 \); if the input of \( f(x) \) is \( g \), this requires

\[
1 < g(x) \leq 8,
\]

This completes the first step.

Step 2: Identify which \( x \) values do not produce values of \( g(x) \) described in step 1. Starting from the inequality, \( 1 < g(x) \leq 8 \), substitute \( g(x) = x^2 + 4 \)

\[
1 < x^2 + 4 \leq 8,
\]

or alternatively by subtracting 4:

\[
-3 < x^2 \leq 4.
\]

However, \( -3 < x^2 \) is always true—\( x^2 \) is always non-negative—and adds no new information.
The non-trivial inequality associated with $-3 < x^2 \leq 4$ is:

$$x^2 \leq 4.$$ 

$x^2 \leq 4$ implies

$$|x| \leq 2.$$ 

We must include absolute value signs since $x^2$ contains no information about $x$’s sign. This finishes step 2, since the $x$ values produce offending values of $g(x)$ in step 1 are those where $|x| > 2$.

**Step 3:** The inequality $|x| \leq 2$ finishes step 2 of the process—identifying which values of $x$ result in a $g(x)$ not in the domain of $f(x)$. However, the result $|x| \leq 2$ was reached only by considering the restriction of the domain due to $f(x)$ alone. To complete step 3, we must consider that $g(x)$ is defined only for $0 \leq x \leq 4$; the actual domain is the intersection of $[-2,2]$ and $[0,4]$—or $[0,2]$.

The full answer is:

$$f(g(x)) = \frac{1}{x^2 - 9} \text{ with domain } 0 \leq x \leq 2.$$

**Glossary:**

This section will define key terms used throughout this handout.

**Combinations:** Combinations are sums, products, differences, quotients, and compositions of unit functions.

**Compositions:** A composite function is one whose input is another function. Composition refers to the process of finding a composite function.

**Unit Functions:** Functions whose properties (e.g. domain, range, and graph) must be memorized. Appendix I lists unit functions.
Appendix I: List of Unit Functions

List of common unit functions where $n$ and $a$ can be any real number that are usually encountered in precalculus:

1. $a$ “Constant function”
2. $x^n$ “Power Function”
3. $a^x$ “Exponential Function”
4. $\log_a x$ “Logarithmic Function”
5. $\sin x$ “Sine Function”
6. $\cos x$ “Cosine Function”
7. $\tan x$ “Tangent Function”
8. $\sin^{-1} x$ “Inverse Sine”
9. $\cos^{-1} x$ “Inverse Cosine”
10. $\tan^{-1} x$ “Inverse Tangent”

These unit functions can be added, subtracted, multiplied, divided, or composed with one another to create combinations. Examples of combinations are given on the next page.

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3 Strictly speaking, sines/cosines are not unit functions as they are combinations of exponentials; however, we will disregard this technicality. Similarly, the other three inverse trigonometric functions (secant, cosecant, and cotangent) differ...
Example A-1: Let \( f(x) = x^3 \), \( g(x) = \frac{1}{x} \) and \( h(x) = 3x \), we can combine these to create \( p(x) = g(f(x)) + h(x) \)

\[
p(x) = g(f(x)) + h(x)
\]

\[
= \frac{1}{f(x)} + 3x
\]

Substitute \( h(x) \) and \( g(x) \)

\[
= \frac{1}{x^3} + 3x
\]

Substitute \( f(x) \)

Conclusion: \( p(x) = \frac{1}{x^3} + 3x \), which is an example of a combination.

Example A-2: The list of unit functions may exclude some functions you are familiar with: for example there is no linear function. To define a linear combination, let:

- \( f(x) = m \) (a constant function with constant \( m \))
- \( g(x) = x \) (a power function evaluated at \( n = 1 \))
- \( h(x) = b \) (a constant function with constant \( b \))

Then we may define a linear function as

\[
(fg + h)(x)
\]

which when evaluated is

\[
(fg + h)(x) = mx + b.
\]

Note on Power Function: For the power function, \( n \) can be any number. One particularly important classes are when \( n = \frac{1}{q} \) since

\[
\frac{1}{x^q} = \sqrt[q]{x}
\]

which reduces to the square root when \( q = 2 \). The other particularly important class are when

\( n = -m \), since

\[
x^{-m} = \frac{1}{x^m}
\]
Practice Problems

1) Let \( f(x) = \frac{x^2}{2} \) and \( g(x) = \frac{2}{2x-4} \).

a) Find \( g(f(x)) \)
2) Let \( f(x) = x^2 + 2 \) with domain \(-2 \leq x \leq 2\), \( g(x) = \frac{x}{x-2} \) with domain \( 3 \leq x \leq 8 \).

a) Find an expression for \( f(g(x)) \)

b) Find the domain of \( f(g(x)) \)