This handout will:

- Discuss the concept of function composition.
- Provide examples of combining multiple functions.
- Provide an example how to find the domain of a function composition.

The reader should be familiar with domain and range of functions.

Motivation for Understanding Composition:

This section will explain why composition functions are important.

Precalculus is the study of functions and their behavior. To understand a wide variety of functions, we study both **unit functions**[1]**,** whose properties must be memorized, and **combinations,** which include sums, differences, products, quotients and **compositions** of unit functions. Combinations are important in calculus, as important rules ("derivative rules") are defined *specifically* only for unit functions and *generically* for combinations**.** Addition, subtraction, products, and quotients correspond to the classical operations for numbers (e.g. the sum of $f(x) = x^2$ and $g(x) = x^3$ is $(f+g)(x) = x^2 + x^3$). Composition, on the other hand, is an operation uniquely defined for functions.

Knowing how to dissect a combination into its unit functions is an equally important skill since calculus rules are defined *generically* for combinations. While most examples will show to compose two functions to illustrate what composition is as an operator, a more important long-term skill is to examine a composition function and **decompose** it into its original functions. Refer to section 1.4 "Decomposing Functions" in Axler's *Precalculus: A Prelude to Calculus,* as this handout does not discuss function decomposition.

[1] See Appendix 1 for definition of unit functions. Note that the term **unit functions** is defined only for the purposes of this handout and the definition is unlikely to be found in any precalculus textbook.

Definition and Notation of Composite Functions:

This section will:

- Define Function Composition
- Show Notations for Composition

Definition: A composite function is a function whose input is another function. A composition is computed by replacing the x variable of one function with the new input function $g(x)$. For example, if $f(x) = \sqrt{x}$ and $g(x) = 1 + x^2$, then $f(g(x)) = \sqrt{1 + x^2}$ is a composition of $\sqrt{x}$ and $1 + x^2$—the input of $f(x)$ is another function $g(x)$. Examples in the next section show types of composite functions you may encounter. The term composition refers to the act of composing two functions to make a composite function.

Notation: There are two canonical notations used for composition. Let $h(x)$ be a composition of $f(x)$ and $g(x)$, then we may write

$$h(x) = f(g(x))$$

or

$$h(x) = (f \circ g)(x).$$

In either case, we pronounce the composition as "f" of "g". Each notation has different emphases: The $(f \circ g)(x)$ notation emphasizes the combination nature of compositions, e.g. by following the same form as sums/differences, $(f \pm g)(x)$; $f(g(x))$ emphasizes the mathematical procedure of substituting $g(x)$ in place of x in the function $f(x)$. We will primarily use the $f(g(x))$ notation throughout this handout.

Worked Examples—Composing Functions:

This section will provide worked examples of composing two functions.

Example 1

Let $f(x) = \frac{1}{x} + 2$ and $g(x) = x^2$, find $f(g(x))$.

$$f(g(x)) = f(g(x))$$

$$= \frac{1}{g(x)} + 2 \qquad \text{Substitute } g(x) \text{ for } x \text{ in } f(x)$$ [2]

$$= \frac{1}{(x^2)} + 2 \qquad \text{Substitute in } g(x)$$

$$= \frac{1}{x^2} + 2 \qquad \text{Simplify}$$

Conclusion: $f(g(x)) = \frac{1}{x^2} + 2$. The above method outlines a typical calculation used to compute composition functions.

Example 2

Let $f(x)$ and $g(x)$ be the same as in example 1, but instead of finding $f(g(x))$, find $g(f(x))$.

$$g(f(x)) = g(f(x))$$

$$= (f(x))^2 \qquad \text{Substitute } f(x) \text{ for } x \text{ in } g(x)$$

$$= \left(\frac{1}{x} + 2\right)^2 \qquad \text{Substitute in } f(x)$$

$$= \frac{1}{x^2} + \frac{2}{x} + \frac{2}{x} + 4 \qquad \text{Expand}$$

$$= \frac{1}{x^2} + \frac{4}{x} + 4 \qquad \text{Simplify}$$

Conclusion: $g(f(x)) = \frac{1}{x^2} + \frac{4}{x} + 4$ in this example and $f\big(g(x)\big) = \frac{1}{x^2} + 2$: $f\big(g(x)\big) \neq g\big(f(x)\big)$, similar to how $4 - 3 \neq 3 - 4$ and $\frac{2}{3} \neq \frac{3}{2}$.

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[2] The phrase "substitute $g(x)$ for x in $f(x)$" means replace the x found in expression $f(x)$ with $g(x)$. Future examples will use this phrase frequently.

Example 3

Let $f(x) = x^2 + x$ and $g(x) = 1 + x$, find $f(g(x))$

$$
\begin{aligned}
f(g(x)) &= f(g(x)) && \\
&= (g(x))^2 + g(x) && \text{Substitute } g(x) \text{ for } x \text{ in } f(x) \\
&= (1 + x)^2 + (1 + x) && \text{Substitute in } g(x) \\
&= 1 + 2x + x^2 + 1 + x && \text{Expand} \\
&= x^2 + 3x + 2 && \text{Simplify}
\end{aligned}
$$

Conclusion: $f(g(x)) = x^2 + 3x + 2$.

<u>Composing of More Than Two Functions:</u>

This section will:

- Define composition of more than two functions
- Provide an example thereof.

We sometimes are faced with the problem of finding the composition more than two functions,

$$f \circ g \circ h \equiv f\Big(g\big(h(x)\big)\Big).$$

To find the composition, we first substitute $g(h(x))$ for x in $f(x)$, and then substitute $h(x)$ for x in $g(x)$, and then substitute in $h(x)$. Example 4 on the next page provides an example of this process.

Example 4

Let $f(x) = x - 10$, $g(x) = x^2$, and $h(x) = x - 2$, find $f(g(h(x)))$

$f(g(h(x)))$	$= f(g(h(x)))$	
	$= g(h(x)) - 10$	Substitute $g(h(x))$ for x in $f(x)$
	$= (h(x))^2 - 10$	Substitute $h(x)$ for x in $g(x)$
	$= (x-2)^2 - 10$	Substitute equation in for $h(x)$
	$= x^2 - 4x + 4 - 10$	Expand equation
	$= x^2 - 4x - 6$	Simplify equation

Conclusion: $f(g(h(x))) = x^2 - 4x - 6$.

Domains of Composite Functions:

This section will:

- Outline a three-step process to find domain of functions
- Provide an example using this three-step process.

Formally, the domain of $f(g(x))$ is the set of real numbers that are in the domain $g(x)$ and satisfy the domain of $f(g(x))$. A three-step process for finding the domain is:

Step 1. First identify what value(s) of $g(x)$ are *not* in the domain of $f(x)$.

Step 2. Find what value(s) of x do not produce the values of $g(x)$ described in step 1.

Step 3. The domain of $f(g(x))$ is the domain of $g(x)$ excluding the x values found in the previous step ***and*** excluding x values not in $g(x)$'s domain.

Example 5

Let $f(x) = \frac{1}{x-13}$ with domain $1 < x \leq 8$ and $g(x) = x^2 + 4$ with domain $0 \leq x \leq 4$. Find the composition and domain of $f(g(x))$

$f(g(x)) = \frac{1}{g(x)-13}$	Substitute $g(x)$ for x in $f(x)$
$= \frac{1}{(x^2+4)-13}$	Substitute in $g(x)$
$= \frac{1}{x^2+4-5}$	Clear parentheses
$= \frac{1}{x^2-9}$	Simplify denominator

Although we found the expression $f(g(x)) = \frac{1}{x^2-9}$, the problem is not complete until we have determined the domain. We will use the three-step process outlined on the previous page.

Step 1: The first step is to identify what values of $g(x)$ are not in the domain of $f(x)$. Since $f(x)$ has a domain $1 < x \leq 8$; if the input of $f(x)$ is g, this requires

$$1 < g(x) \leq 8,$$

This completes the first step.

Step 2: Identify which x values do not produce values of $g(x)$ described in step 1. Starting from the inequality, $1 < g(x) \leq 8$, substitute in $g(x) = x^2 + 4$

$$1 < x^2 + 4 \leq 8,$$

or alternatively by subtracting 4:

$$-3 < x^2 \leq 4.$$

However, $-3 < x^2$ is always true—$x^2$is always non-negative—and adds no new information.

The non-trivial inequality associated with $-3 < x^2 \leq 4$ is:

$$x^2 \leq 4.$$

$x^2 \leq 4$ implies

$$|x| \leq 2.$$

We must include absolute value signs since x^2 contains no information about x's sign. This finishes step 2, since the x values produce offending values of $g(x)$ in step 1 are those where $|x| > 2$.

Step 3: The inequality $|x| \leq 2$ finishes step 2 of the process—identifying which values of x result in a $g(x)$ not in the domain of $f(x)$. However, the result $|x| \leq 2$ was reached only by considering the restriction of the domain due to $f(x)$ alone. To complete step 3, we must consider that $g(x)$ is defined only for $0 \leq x \leq 4$; the actual domain is the intersection of $[-2,2]$ and $[0,4]$—or $[0,2]$.

The full answer is:

$$f(g(x)) = \frac{1}{x^2 - 9} \text{ with domain } 0 \leq x \leq 2.$$

Glossary:

This section will define key terms used throughout this handout.

Combinations: Combinations are sums, products, differences, quotients, and compositions of unit functions.

Compositions: A composite function is one whose input is another function. Composition refers to the process of finding a composite function.

Unit Functions: Functions whose properties (e.g. domain, range, and graph) must be memorized. Appendix I lists unit functions.

Appendix I: List of Unit Functions

List of common unit functions where n and a can be any real number that are usually encountered in precalculus:

1. a "Constant function"
2. x^n "Power Function"
3. a^x "Exponential Function"
4. $\log_a x$ "Logarithmic Function"
5. $\sin x$ "Sine Function"[3]
6. $\cos x$ "Cosine Function"
7. $\tan x$ "Tangent Function"
8. $\sin^{-1} x$ "Inverse Sine"
9. $\cos^{-1} x$ "Inverse Cosine"
10. $\tan^{-1} x$ "Inverse Tangent"

These unit functions can be added, subtracted, multiplied, divided, or composed with one another to create combinations. Examples of combinations are given on the next page.

[3] Strictly speaking, sines/cosines are *not* unit functions as they are combinations of exponentials; however, we will disregard this technicality. Similarly, the other three inverse trigonometric functions (secant, cosecant, and cotangent) differ

Example A-1: Let $f(x) = x^3$, $g(x) = \frac{1}{x}$ and $h(x) = 3x$, we can combine these to create $p(x) = g\big(f(x)\big) + h(x)$

$$p(x) = g\big(f(x)\big) + h(x)$$

$$= \frac{1}{f(x)} + 3x \qquad \text{Substitute } h(x) \text{ and } g(x)$$

$$= \frac{1}{x^3} + 3x \qquad \text{Substitute } f(x)$$

Conclusion: $p(x) = \frac{1}{x^3} + 3x$, which is an example of a combination.

Example A-2: The list of unit functions may exclude some functions are you are familiar with: for example there is no linear function. To define a linear combination, let:

- $f(x) = m$ (a constant function with constant m))
- $g(x) = x$ (a power function evaluated at $n = 1$)
- $h(x) = b$ (a constant function with constant b)

Then we may define a linear function as

$$(fg + h)(x)$$

which when evaluated is

$$(fg + h)(x) = mx + b.$$

Note on Power Function: For the power function, n can be *any* number. One particularly important classes are when $n = \frac{1}{q}$, since

$$x^{\frac{1}{q}} = \sqrt[q]{x}$$

which reduces to the square root when $q = 2$. The other particularly important class are when $n = -m$, since

$$x^{(-m)} = \frac{1}{x^m}.$$

Practice Problems

1) Let $f(x) = \frac{x^2}{2}$ and $g(x) = \frac{2}{2x-4}$.

   a) Find $g(f(x))$

2) Let $f(x) = x^2 + 2$ with domain $-2 \leq x \leq 2$, $g(x) = \frac{x}{x-2}$ with domain $3 \leq x \leq 8$.

a) Find an expression for $f(g(x))$

b) Find the domain of $f(g(x))$