Campus Academic Resource Program

Double Angle and Half Angle Formulae

This handout is an overview of double angle and half angle formulae found in pre-calculus. In particular, this handout will:

- Discuss a strategy to find an expression for the **double-angle formula** for cosine. That is, we will discuss a geometric way to show $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
- Briefly present the **half-angle formulae.**
- Provide an example to illustrate *how* a **double-angle formula** can be used. In particular, emphasis will be placed on how the **double-angle formulae** and **half-angle formulae**, despite their names, can be used for more than doubling and halving angles.

Students who are utilizing this handout should already be familiar with these following topics:

- The unit circle and radians.
- How sine and cosine are defined within the unit circle.
- The Pythagorean Identity as it's presented in trigonometry.

Note that this handout will use the convention that given a line segment adjoining two points A and B:

- $\overline{AB}$ represents the line segment adjoining the two points
- $|\overline{AB}|$ represents the *length* of a line segment adjoining A and B. That is, $|\overline{AB}|$ is the distance between A and B.

This handout will use the convention for a triangle with vertices at point, A, B **,** and C (see Appendix I for more information):

- $\Delta\mathrm{ABC}$ refers to the name of the triangle
- $\angle ABC$ refers to the angle opposite of the length $\overline{AC}$; that is the angle nearest the point "B."

Strategy for Finding the Double-Angle Formula for Cosine[1]:

The **double-angle formula** for cosine is the expression that says:

$$\boxed{\cos 2\theta = \cos^2\theta - \sin^2\theta}$$

This expression is found by evaluating two triangles, with one triangle stacked on the other, each with an angle θ. See figure 1 for a visual. In the unit circle, our triangles, ΔFAB and ΔFBC, would have the configuration seen in figure 1. The configuration seen in figure 1 has these important properties:

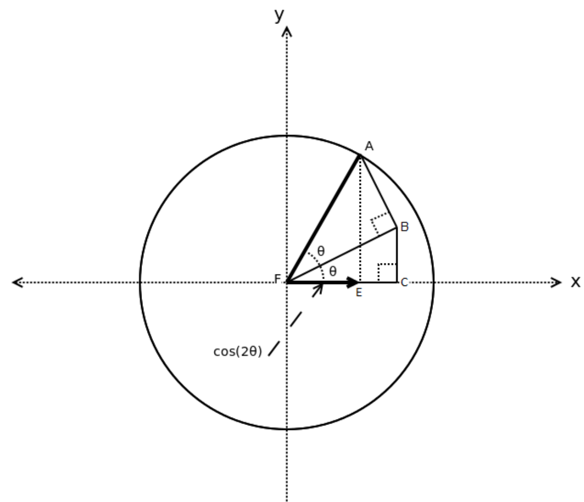

Figure 1: *This figure shows our configuration made up of two right triangles. One right triangle ΔFAB is on top of another right triangle ΔFBC.*

- The line $\overline{AF}$ has a length of 1, as it's a radius in the unit circle.

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- Note that ΔAFE is a right triangle. The angle $\angle AFE$ has an angle of 2θ. We can then use the definition of cosine, $\text{cosine} = \frac{\text{adjacent}}{\text{hypotenuse}}$, to obtain that $\cos 2\theta = \frac{|\overline{FE}|}{|\overline{FA}|}$.

- Since the circle in figure 1 is a unit circle, the length $\left|\overline{FA}\right| = 1$ as it's the radius of the unit circle. Therefore, $\frac{|\overline{FE}|}{|\overline{FA}|} = \frac{|\overline{FE}|}{1} = |\overline{FE}|$. Thus, because $\cos 2\theta = \frac{|\overline{FE}|}{|\overline{FA}|}$ and $\frac{|\overline{FE}|}{|\overline{FA}|} = |\overline{FE}|$, we can conclude that $\cos 2\theta = |\overline{FE}|$

- Thus finding the length $\overline{FE}$ determines (i.e. is equal) to find the expression for $\cos 2\theta$.

Once we have accepted that $|\overline{FE}| = \cos 2\theta$, we can then analyze the configuration to figure out another expression for the length $\overline{FE}$ in terms of $\cos\theta$ and $\sin\theta$. We need to analyze the geometry in order to find this expression for $|\overline{FE}| = \cos 2\theta$; which is the process on the next page.

[1] If you would prefer a strategy that requires less lines and are comfortable with complex numbers, please refer to the "Euler's Formula and Trigonometry" worksheet for an alternative derivation.

The steps below summarize the essential parts to find $\cos 2\theta$, or the length of $\overline{FE}$, referencing the triangle ΔAFE seen in figure 2. Read each step carefully, as each step contains a lot of information.

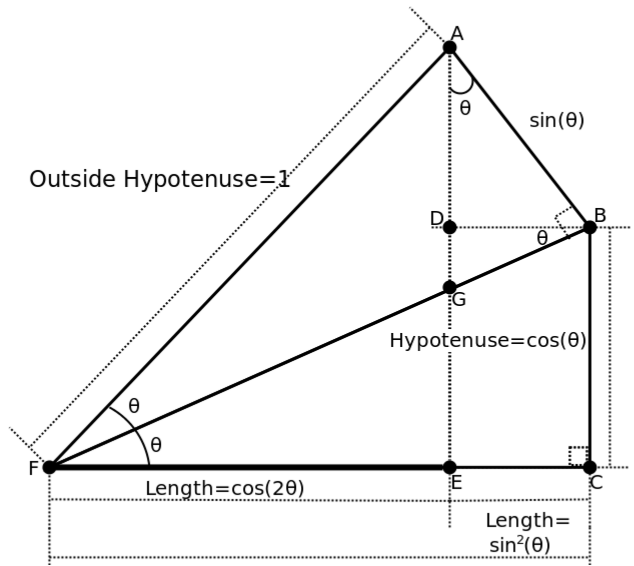


Figure 2: *The figure shows the triangle ΔAFE. Here, all relevant lengths and angles referenced in the bullet points to the left are labeled.*

a) We know that the **hypotenuse** of ΔAFE is 1, as we interpret the length $\overline{FE}$ to be a radius in the unit circle, i.e. the triangle in figure 1.

b) The **hypotenuse** of ΔFBC, the length $\overline{FB}$, is $1 \cdot \cos\theta$, as it is the **adjacent** length of the triangle ΔFAB.

c) The **adjacent** length of the triangle ΔFBC is $\cos\theta = \frac{|\overline{FC}|}{|\overline{FB}|}$, or $\cos\theta = \frac{|\overline{FC}|}{\cos\theta}$. This means that the adjacent length $|\overline{FC}| = \cos^2\theta$.

d) The length of $\overline{FE}$, which is the length we're interested in, is the length of $\overline{FC}$ minus the length of $\overline{CE}$. We know the length of $\overline{FC}$ is $\cos^2\theta$ (see step c). Thus, we can find the length of $\overline{FE}$, which is what we're after, if we know the length of $\overline{CE}$.

e) $\overline{CE}$ is the same length as $\overline{DB}$. The segment $\overline{DB}$ is the side of the triangle ΔADB opposite to the angle $\angle DAB$. The triangle ΔADB has a hypotenuse of $\sin\theta$.

f) Since the triangle ΔADB has a **hypotenuse** of $\sin\theta$. Thus, we can use the fact $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{|\overline{DB}|}{|\overline{AB}|}$ to show that $\overline{DB}$ has a length of $\sin^2\theta$.

g) Since $\overline{CE}$ the same length as $\overline{DB}$, $\overline{CE}$ has a length of $\sin^2\theta$.

h) The overall length of $\overline{FE}$, which is $\cos 2\theta$, is the difference of the lengths $\overline{FC}$ and $\overline{CE}$ which is:

$$|\overline{FE}| = |\overline{FC}| - |\overline{CE}|$$

$$\boxed{\cos(2\theta) = \cos^2\theta - \sin^2\theta}$$

The Half Angle Formulae:

Now that we have the double-angle formula for cosine, we can use the Pythagorean identity to find two more identities known as the half-angle formulae:

$$\boxed{\cos^2\theta = \frac{1+cos(2\theta)}{2}} \qquad \boxed{\sin^2\theta = \frac{1-cos(2\theta)}{2}}$$

Although their name is "half angle formulae," their purpose in mathematics is often to rewrite expressions involving $\sin^2\theta$ and $\cos^2\theta$ without squaring any trigonometric functions. As such, you must use the substitution $\theta = \frac{\varphi}{2}$ to obtain an expression that is useful for halving angles.

Applying Half-Angle and Double Angle Formula:

The main purpose of the double angle formula and half angle formula are to enable us to simplify trigonometric expressions. For example, suppose we had to solve the equation:

$$(\cos\theta + sin\theta)(\cos\theta - \sin\theta) = \frac{1}{2}$$

We can factor out $(\cos\theta + sin\theta)(\cos\theta - \sin\theta)$ (e.g. using FOIL) to show that:

$$\cos^2\theta - \sin^2\theta = \frac{1}{2}$$

Next, we can use the double-angle formula to rewrite $\cos^2\theta - \sin^2\theta = \cos 2\theta$, allowing us so simplify the expression as[2]:

$$\cos 2\theta = \frac{1}{2}$$

[2] In this particular example, you do *not* need to use the double angle formula for cosine. However, for the sake of an example, we will use it here. The other alternative would be to use the Pythagorean Identity.

If we simplify even further by applying the arccosine function to both sides, we obtain:

$$2\theta = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

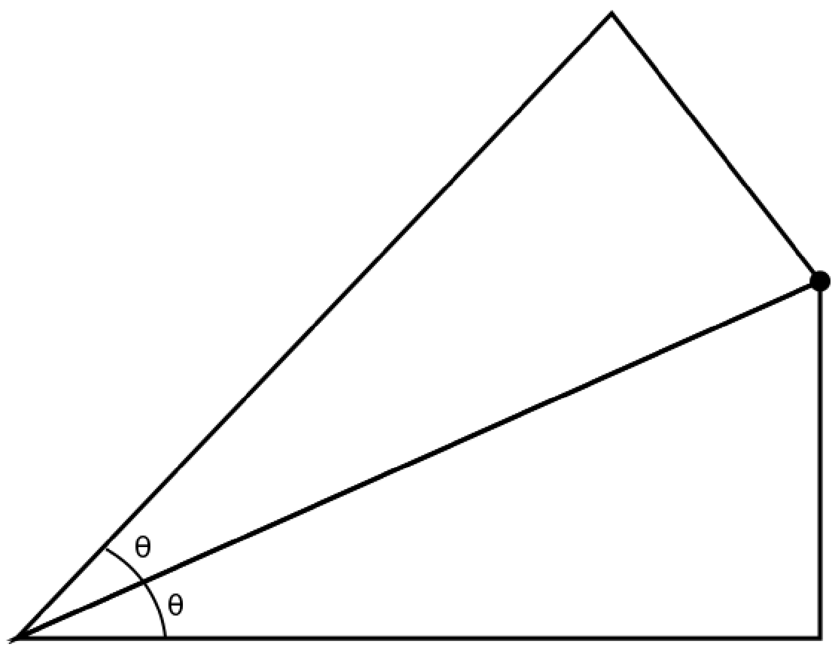
And if we divide both sides of the equation above by 2, we can find:

$$\theta = \frac{\pi}{6}$$

While this particular example may seem contrived—that is to say, an expression such as $(\cos\theta + sin\theta)(\cos\theta - \sin\theta)$ may not seem like an expression that would apply in real-world equations—many such expressions show up naturally when solving a calculus II expression known as an "integral." Such calculus expressions often require us to use trigonometry to solve them, and moreover, in order to simplify the expressions we *must* use the half angle and double angle formulae.

Practice Problems:

1. Show that $\mathbf{\sin(2\theta) = 2\sin\theta \cdot \cos\theta.}$ This proof should share some similarities to finding $\cos 2\theta$. Start by identifying $\sin(2\theta)$ and figure out what length(s) and angle(s) you need to find in order to solve for $\sin(2\theta)$



2. Derive the identities $\sin^2\theta = \frac{1-cos(2\theta)}{2}$ and $\cos^2\theta = \frac{1+cos(2\theta)}{2}$ by using the double angle formula for cosine and the Pythagorean Identity.

3. Use the double angle formula for sine to find one exact angle such that $2\sin\theta\cos\theta = 1$.

4. Use the double angle formula for cosine to find one exact angle such that

$$\sin^2\theta - \cos^2\theta = \frac{\sqrt{3}}{2}$$

5. Find an expression for $\tan 2\theta$ in terms of $\tan\theta$.

6. *Open ended question:* Look at the half angle and double angle formula: explain *why* these formulae might be useful beyond doubling angles. That is, why might we prefer one side of the equation over the other side.

Glossary:

Adjacent Side: An adjacent side of a right triangle refers to the side that is adjacent (i.e. next to) to a given angle and adjacent to the right angle. See "Appendix 2" for more information.

Double Angle Formula: Abstractly, the double angle formulae are expressions equivalent to $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$ [3]. Namely, $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = \cos^2\theta - \sin^2\theta$, and $\tan 2\theta = \frac{2\tan\theta}{1-\tan 2\theta}$.

Half Angle Formulae: Half angle formulae refer to expressions that are equivalent to $\sin\left(\frac{\phi}{2}\right)$ and $\cos\left(\frac{\phi}{2}\right)$ (as well as other trigonometric functions). Typically they are presented as $\cos^2\theta = \frac{1+cos(2\theta)}{2}$ and $\sin^2\theta = \frac{1-cos(2\theta)}{2}$.

Hypotenuse: In a right triangle, the longest length is called the hypotenuse and is always opposite of the right angle. See "Appendix 2" for more information.

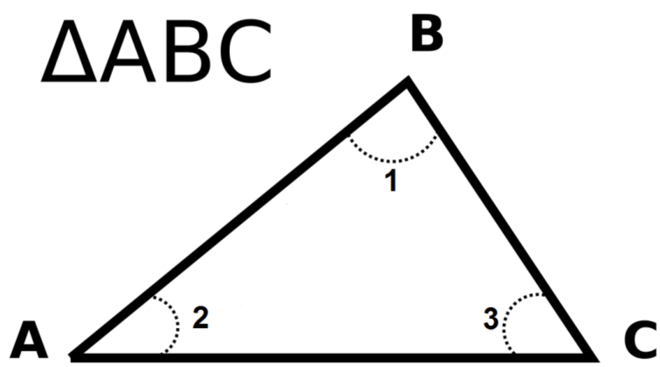
Opposite Side: An opposite side of a right triangle refers to the side that is opposite to a given angle and adjacent to the right angle for visual. See "Appendix 2" for more information.

Pythagorean Identity: The Pythagorean Identity is the statement $\sin^2\theta + \cos^2\theta = 1$. See the Pythagorean Identity worksheet for more information.

[3] There are also double angle-formulae for the other three trigonometric functions (**cotangent, secant, and cosecant);** however, we will not concern ourselves with them here.

Appendix I: Notation for Triangles

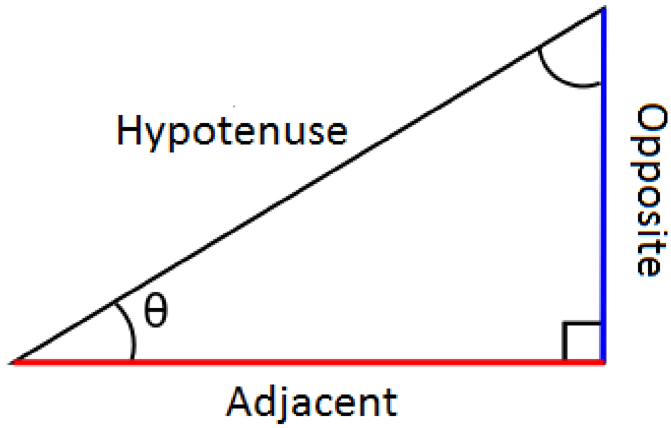
The figure below shows a triangle with vertices labeled ABC. This appendix uses the word "vertex" to refer to a point where two lines meet. In this case, the points *A, B,* and *C* are the vertices.



- The *name* of the triangle should be denoted first by a "Δ" and then listing off the three vertices in any order. For the triangle to the right, either Δ*ABC,* Δ*ACB,* Δ*BAC,* Δ*BCA,* Δ*CAB,* Δ*CBA* are all acceptable notations.
- The *angle* of an angle in a triangle should first be denoted by a "∠" and then listing off the three vertices; however, the second *letter* should of the angle should correspond to that angle's vertex. For example, referencing the triangle to the right, these are acceptable names for:
  - Angle 1: Either $\angle ABC$ and $\angle CBA$ are acceptable. Second letter is *B.*
  - Angle 2: Either $\angle BAC$ and $\angle CAB$ are acceptable. Second letter is *A.*
  - Angle 3: Either $\angle BCA$ and $\angle ACB$ are acceptable. Second letter is *C.*

  In each case, the second letter is the same as the vertex nearest that particular angle.

Appendix II: Defining Adjacent and Opposite Sides.



Frequently when discussing *right* triangles, we speak of "adjacent" and "opposite" sides. Figure 4 to the right summarizes the definitions of adjacent and opposite side. A more detailed definition is as follows:

- The side adjacent to a given angle (θ in the figure) is the leg *closer* ("adjacent") to that angle.
- The side opposite of a given angle (θ in the figure) is the leg *opposite* to that angle.
    - A "leg" refers to a side of a right triangle that is not the hypotenuse (i.e. not the longest side of the right triangle).

In both cases, adjacent and opposite sides must be defined with respect to a particular angle. For this handout, reference angles are explicitly listed in the figures corresponding a discussion. Thus, if we discuss an "adjacent" side with respect to a "certain triangle," the figure will indicate what angle we are using as reference.

Two useful relationships involving the lengths of the triangle, which follow from the triangular definition of sine and cosine, are as follows:

- $\boxed{\text{Opposite} = \text{Hypotenuse} \cdot \sin\theta}$
- $\boxed{\text{Adjacent} = \text{Hypotenuse} \cdot \cos\theta}$

References:

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