This handout will:

- Distinguish between the Pythagorean *Theorem* and Pythagorean *Identity*.
- Introduce the Pythagorean Identity mathematically and geometrically.
- Derive the Pythagorean Identity geometrically.
- Apply the Pythagorean Identity in the context of solving algebra problems.

Pythagorean Identity in Trigonometry

One of the most important trigonometric identities is the **Pythagorean Identity**, which is closely related to, and can be derived from, the **Pythagorean Theorem**. If you are unfamiliar with the Pythagorean Theorem, please reference CARP's Pythagorean Theorem Handout for Math 60 here[1]. In order to demonstrate the Pythagorean Identity, we will use the definition of **sine** and **cosine** as defined by the unit circle. If you are unfamiliar with the unit circle, please reference your textbook: for those in Math 199, please refer to the chapter titled "The Unit Circle"; for those in Math 226, 227, or 288, please refer to the chapter titled "Trigonometric Functions" near the beginning of the book.

Students who are using this handout should be familiar with this content:

- The unit circle and radians.
- How sine and cosine are defined on the unit circle.
- The Pythagorean Theorem.
- Definition of tangent, cotangent, secant, and cosecant in terms of sine and cosine (the glossary on page 10 defines these functions).

[1] https://sites7.sfsu.edu/sites/sites7.sfsu.edu.carp/files/PDF/Math/Pre-Calculus-MATH199/The%20Pythagorean%20Theorem%20.pdf

Terminology of Pythagorean Identity vs. Pythagorean Theorem:

This worksheet discusses the Pythagorean Identity. The Pythagorean Identity refers specifically to the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$. The Pythagorean Theorem refers to the theorem that states that $(\text{length}_1)^2 + (\text{length}_2)^2 = \text{hypotenuse}^2$. The two are related (in that we need to use one to show the other), but they nevertheless refer to different identities.

Introduction to Pythagorean Identity:

This section will…

- Define the Pythagorean Identity
- Derive the Pythagorean Identity

The Pythagorean Identity is an **identity** that relates sine and cosine mathematically. In order to express this relationship, let us consider a radius in the **unit circle** that makes an angle θ with the positive x-axis. The horizontal line in figure 1 is defined to be $\cos\theta$ and the vertical line in figure 1 is defined to be $\sin\theta$.

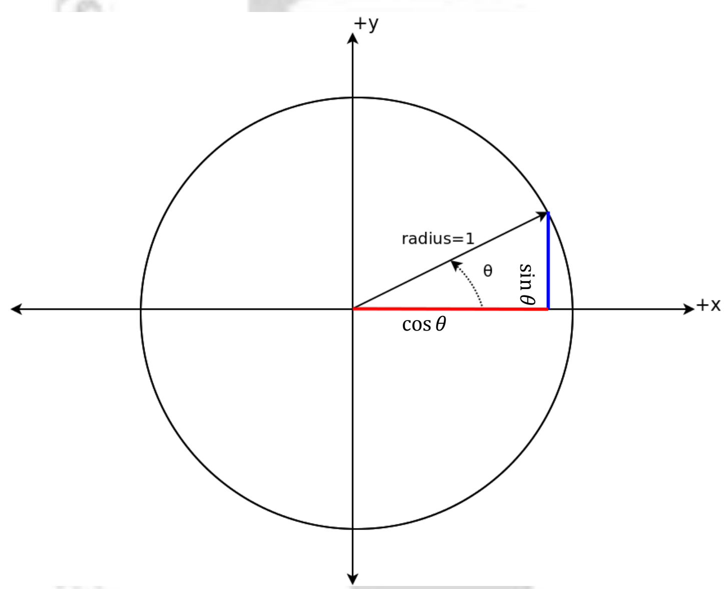

Figure 1: This is a plot of sine and cosine on the unit circle. The horizontal length is cosine, and the vertical length is sine.

If we view the radius, horizontal red line, and vertical blue line as a triangle, then we have the triangle seen in Figure 2 below. The hypotenuse in Figure 2 coincides with the radius in Figure 1; since every radius in the unit circle is 1, the hypotenuse also has a length of 1. By the unit circle definition of sine and cosine, the horizontal length is $\cos\theta$ and the vertical length is $\sin\theta$. Figure 2 summarizes all relevant information.

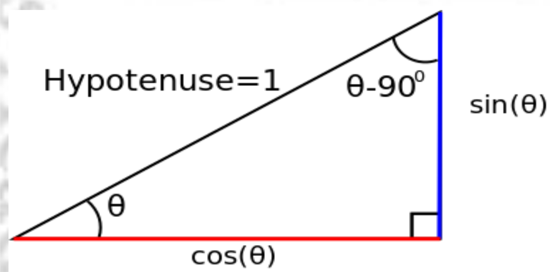

Figure 2: This triangle is an extraction of the triangle drawn on the unit circle in figure 1.

We may use the use the Pythagorean Theorem,

$$(\text{Opposite})^2 + (\text{Adjacent})^2 = \text{Hypotenuse}^2,$$

to find

$$(\sin\theta)^2 + (\cos\theta)^2 = \text{Hypotenuse}^2$$

However, by the definition of the unit circle, we take the hypotenuse of this triangle to be 1, so:

$(\sin\theta)^2 + (\cos\theta)^2 = 1$	**Pythagorean Identity**

which is the Pythagorean Identity.

Using the Pythagorean Identity to Solve Problems:

This section will describe how to solve specific algebraic problems using the Pythagorean Identity.

The Pythagorean Theorem can be used to solve for $\sin\theta$ and $\cos\theta$ explicitly if we know the value of any other trigonometric function. The example on the following page will use cotangent.

Example: Suppose $\cot\theta = 2$, what is $\sin\theta$?

Solution: Start by writing the definition of cotangent:

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Substitute in $\cot\theta = 2$:

$$2 = \frac{\cos\theta}{\sin\theta}$$

Find a relationship between sine and cosine:

$$2\sin\theta = \cos\theta$$

Write the Pythagorean Identity:

$$(\sin\theta)^2 + (\cos\theta)^2 = 1.$$

Substitute in $\cos\theta = 2\sin\theta$:

$$(\sin\theta)^2 + (2\sin\theta)^2 = 1$$

Simplify:

$$\sin^2\theta + 4\sin^2\theta = 1$$

Add like terms:

$$5\sin^2\theta = 1$$

Divide both sides by 5

$$\sin^2\theta = \frac{1}{5}$$

Square root both sides. The square root results in two solutions, one positive and one negative.

$$\sin\theta = \pm\frac{1}{\sqrt{5}}$$

Therefore, if $\cot\theta = 2$, then we may conclude that $\sin\theta$ is either $\frac{1}{\sqrt{5}}$ or $-\frac{1}{\sqrt{5}}$.

In general, you want to use the given information to find a relationship between sine and cosine. In the previous example, $\cos\theta = 2\sin\theta$ was that relationship. After you find that relationship, use the Pythagorean Theorem to determine sine or cosine. Problems 4 and 5 provide a similar examples using tangent and secant respectively.

Practice Problems:

1. The triangular definitions of sine and cosine are, respectively, $\sin\theta \equiv \frac{\text{opposite}}{\text{hypotenuse}}$ and $\cos\theta \equiv \frac{\text{adjacent}}{\text{hypotenuse}}$, where the hypotenuse can be of any length. Re-derive the Pythagorean Identity using the triangular definitions for sine and cosine (hint: start by using the Pythagorean Theorem).

2. Using the Pythagorean Identity, show that $1 + \tan^2\theta = \sec^2\theta$ where $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sec\theta = \frac{1}{\cos\theta}$

3. Using the Pythagorean Identity, show that $1 + \cot^2\theta = \csc^2\theta$ where $\cot\theta = \frac{\cos\theta}{\sin\theta}$ and $\csc\theta = \frac{1}{\sin\theta}$

4. Suppose $\tan\theta = 5$. Find two values of $\sin\theta$ exactly using the Pythagorean Theorem.

5. Suppose $\sec\theta = 3$, find two values of $\sin\theta$ exactly using the Pythagorean Identity.

6. **Challenge Problem:** The kinetic energy for a mass on a spring is $\frac{k}{2}A^2\sin^2\theta$ [2]. The potential energy of the same mass on a spring is $\frac{k}{2}A^2\cos^2\theta$. Show that the total energy (the kinetic energy plus potential energy) for a mass on a spring is constant. If $A = 5$ and $k = 2$, what is the total energy?

[2] The "angle" in this problem is nothing more than a mathematical convenience, and is related to how often the mass on a spring goes back and forth. One way to interpret the *cosine* of the angle is a measure of how far the mass has strayed from its equilibrium position.

7. **Challenge Problem:** Someone offers this proof "proving" $\sin(-\theta) = \sin(\theta)$ (***<u>this statement is false</u>***) using the Pythagorean Identity. Their proof goes as follows

a) $\sin^2(-\theta) + \cos^2(-\theta) = 1$ because the Pythagorean Identity always holds

b) $\cos(-\theta) = cos(\theta)$ because cosine is an even function (*this statement itself is true)*

c) $\sin^2(-\theta) = 1 - \cos^2(-\theta) \Longrightarrow \sin(-\theta) = \sqrt{1 - \cos^2(-\theta)} = \sqrt{1 - \cos^2(\theta)}$ by solving for $\sin(-\theta)$ and substituting in $\cos(-\theta) = cos(\theta)$

d) From the Pythagorean Identity, $\sin^2(\theta) + \cos^2(\theta) = 1$ such that $\sin^2(\theta) = 1 - \cos^2(\theta)$, therefore $\sqrt{1 - \cos^2(\theta)} = \sqrt{\sin^2 \theta}$

e) If so, then $\sin(-\theta) = \sqrt{1 - \cos^2(\theta)} = \sqrt{\sin^2 \theta} = \sin \theta$

f) Therefore $\sin(-\theta) = \sin(\theta)$ (again, it must be emphasized this conclusion is *wrong*).

Explain where this proof erred. If you're stuck, try working out this "proof" step by step and carrying out all operations. If you're still stuck, note that the correct identity is $\sin(-\theta) = -\sin(\theta)$; ponder where in the proof we might have lost a minus sign. Explain what there is to learn from this error.

Glossary:

Cosine: In a triangle, cosine is defined as the quotient of a length adjacent to an angle and the hypotenuse, or symbolically, $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$. In the unit circle, cosine is interpreted as the horizontal length.

Cosecant: Cosecant is defined to be reciprocal of sine; that is $\csc\theta = \frac{1}{\sin\theta}$.

Cotangent: Cotangent is defined to be the quotient of cosine and sine, that is $\cot\theta = \frac{\cos\theta}{\sin\theta}$.

Hypotenuse: The hypotenuse is the longest length in a right triangle.

Identity: An identity is a mathematical statement that is always true.

Pythagorean Theorem: The Pythagorean Theorem says for a right triangle, the two lengths of the triangle and hypotenuse are related by the formula $(\text{length}_1)^2 + (\text{length}_2)^2 = (\text{hypotenuse})^2$.

Pythagorean Identity: The Pythagorean Identity says that sine and cosine are related by the identity $\sin^2\theta + \cos^2\theta = 1$.

Secant: Secant is defined to be the reciprocal of cosine, that is $\sec\theta = \frac{1}{\cos\theta}$.

Sine: In a triangle, sine is defined as the quotient of a length opposite to an angle and the hypotenuse, or symbolically $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$. In the unit circle, sine is interpreted as the *vertical* length.

Tangent: Tangent is the quotient of sine and cosine, that is $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

Unit Circle: A circle of radius 1. Often used in definitions.

References:

Axler, S. (2009). Precalculus: A Prelude to Calculus. *AMC*, *10*, 12.

Thomas, G. B., Weir, M. D., Hass, J., & Giordano, F. R. (2010). *Thomas' Calculus Early Transcendentals*. Pearson.

Weisstein, Eric W. "Pythagorean Theorem." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/PythagoreanTheorem.html