Campus Academic Resource Program

The Pythagorean Theorem

This handout will:

- Examine the properties of length and area for the square.
- Provide a visual interpretation of the Pythagorean Theorem
- Apply the Pythagorean Theorem in various contexts

In 570 BC, on the small Mediterranean island of Samos, a boy named Pythagoras was born. Pythagoras grew up to become a skilled mathematician who helped advance mankind's understanding of geometry. His work resulted in the discovery of a simple, yet elegant geometric theorem. This theorem, known as the Pythagorean Theorem, tells us an interesting relationship between the sides of a right triangle. It states the following:

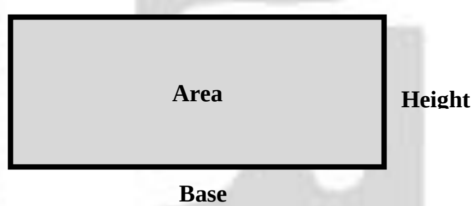
For any right triangle, the square formed by the length of the hypotenuse is equal to the sum of the squares formed by the remaining two sides.

We will be exploring the meaning of this statement by looking at several of the underlying concepts. Once these fundamental concepts are established, we can apply them to understanding the theorem itself. As we will see, the theorem can be thought of in terms of its visual representation. Seeing the visual representation will help you look beyond the equation itself and gain deeper insight into how the theorem works.

Properties of the Square

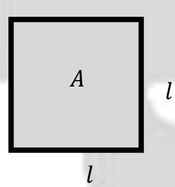
To successfully interpret the Pythagorean Theorem, we must first make sure we are comfortable with the properties of a square. Recall that the area of a rectangular polygon is given by the equation

$$Area = Base \times Height.$$



A square is a special case of rectangle, where each side has the same length. Because of this, a square's base and height can be indicated by a single variable l. This allows our equation for the area (denoted as A) to be written as

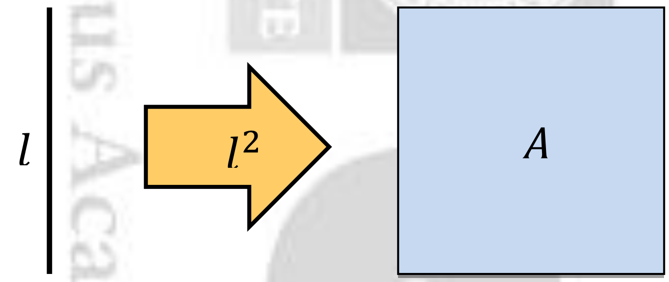
$$A = l \times l.$$



Rather than writing the equation for the square's area in this form, we simplify the notation by introducing the notation for *squaring* a particular quantity. This allows us to write the area of the square as

$$A = l^2$$

This equation can be interpreted by posing the following question: if we are given a line segment of length l, what is the area of the square whose sides are also l? This can be answered by simplify *squaring* l, which is the same as multiplying l by itself.



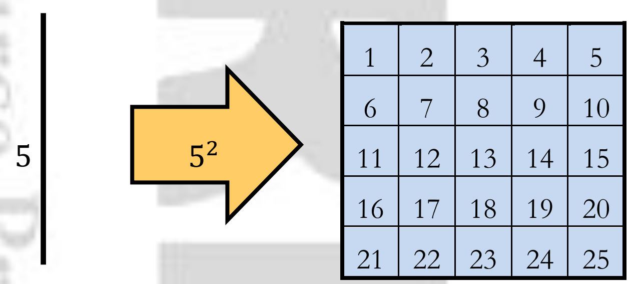
Example 1:

Problem: Evaluate the expression $x = 5^2$ using both arithmetic and visual methods.

Solution: We first want to solve this equation using arithmetic. The equation can either be solved immediately, or we can first express it explicitly and carry out the intermediate steps.

$$x = 5^2$$
$$x = 5 \times 5$$
$$x = 25$$

Thus, our answer is 25. The interpretation of this answer can be seen by considering what it means to square a number. Recall that squaring a number means finding the area of the corresponding square. This interpretation allows us to solve the problem visually.



As we can see, squaring a number gives us a value for the corresponding square's area. What if we wanted to go in the other direction? If we start with an area, how can we find the length of the square's sides? To answer this requires us to first look at our definition of a square's area

$$A = l^2.$$

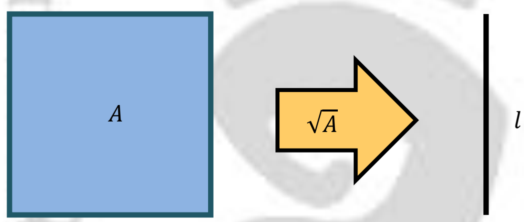
Let us see what happens when we solve the equation for l.

$$A = l^2$$
$$A^{\frac{1}{2}} = (l^2)^{\frac{1}{2}}$$
$$A^{\frac{1}{2}} = l^{(2)\left(\frac{1}{2}\right)}$$
$$A^{\frac{1}{2}} = l$$

Going through these steps has allowed us to isolate our variable l. Because we were able to "undo" the square, this operation is given a special name. It is referred to as the *square root*, and is indicated in the following way:

$$l = A^{\frac{1}{2}} \equiv \sqrt{A}$$

This equation still shows us a relationship between a square's length and area. In this case, we are finding the length of a square's side when we are given the area.



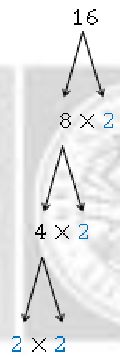
Example 2:

Problem: Evaluate $x = \sqrt{16}$ using both arithmetic and visual methods.

Solution: We first want to use arithmetic to solve for x. You may know the answer offhand, but we can look at a method that can be used to solve the problem explicitly. This method can be used to evaluate or simplify square roots that may be somewhat complicated. The idea is to represent our number as a product of its prime factors. For example, 16 can be expressed as:

$$16 = \underbrace{2 \times 2 \times 2 \times 2}_{\text{Prime factors}}$$

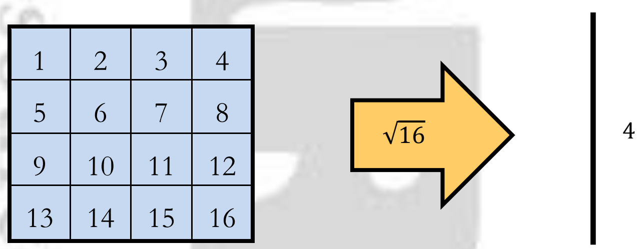
How did we know that 16 could be written this way? We can use the "tree method" to continually divide 16 into whole numbers.



Once the “base” of our tree is composed of only prime factors, we can write our original number as the product of each base. Writing our number this way is useful when evaluating square roots because it allows us to find any square products within our number. By plugging our new representation of the number 16 into the original equation, we have:

$$x = \sqrt{16}$$
$$x = \sqrt{2 \times 2 \times 2 \times 2}$$
$$x = \sqrt{2^2 \times 2^2}$$
$$x = \sqrt{2^2} \times \sqrt{2^2}$$
$$x = 2 \times 2$$
$$x = 4$$

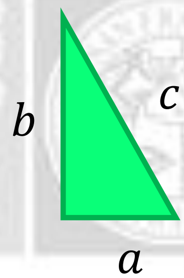
Our answer is 4, but let us see how this answer relates to the visual depiction. This solution tells us that a square of area 16 has sides of length 4.



We now know enough about squares to provide a complete description of the Pythagorean Theorem. Our goal is to now apply this knowledge when dealing with right triangles.

The Pythagorean Theorem

Suppose we were given the following triangle:

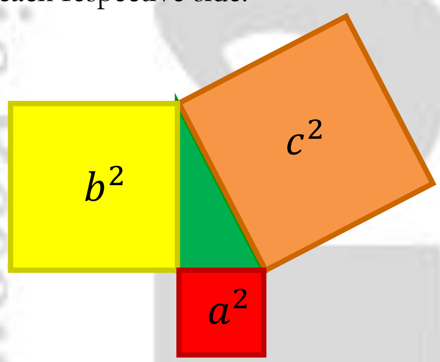


In this case, a, b, and c indicate the lengths of each side. We can motivate our study of the Pythagorean Theorem by asking the following question: "If we know the lengths of both a and b, how can we find the length of c?" Look back at our initial definition of the Pythagorean Theorem given in the introduction. It tells us that there exists a relationship between sides a, b, and c given by the equation:

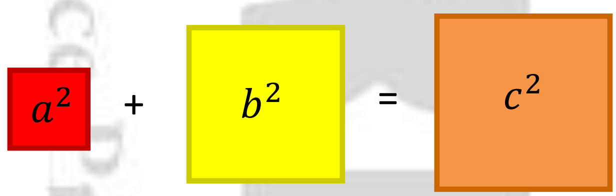
Pythagorean Theorem: $a^2 + b^2 = c^2$

By using this relationship, we will be able to find our length c.

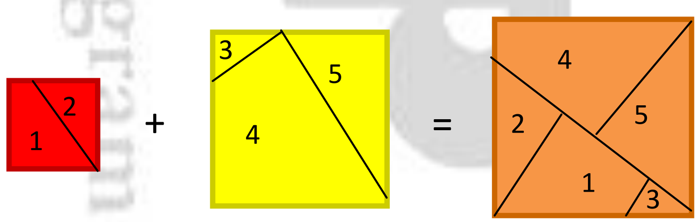
Before we look into finding our length c, let us first represent the theorem visually. We can draw a^2, b^2, and c^2 as the squares of each respective side.



According to the theorem, the area of a^2 added with the area of b^2 should be the same as the area of c^2. In other words,



To help convince ourselves that this is true, we divide a^2 and b^2 in the following way:



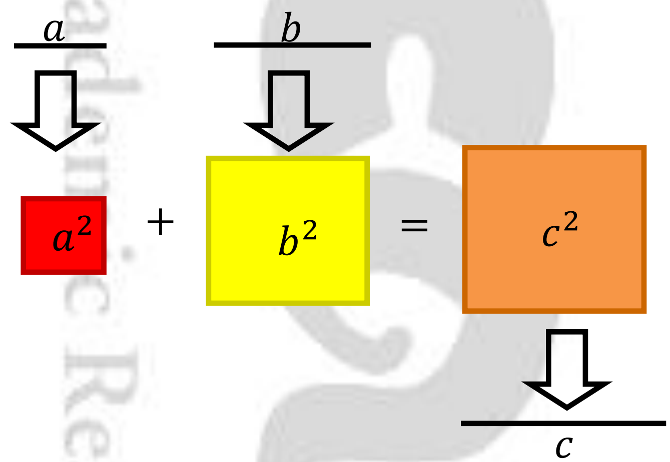
Be aware that this demonstration does not provide a formal geometric proof, but rather a visual demonstration of how the theorem works.

We would now like to use the theorem in order to find the length of the hypotenuse. Because c^2 can be represented in terms of $a^2$and b^2, we can take the square root to find our desired length c.

The logic behind this process is:

- We know the lengths a and b, but not the length c.
- The square formed by length c can be expressed using the sum of squares formed by a and b.
- Once c^2 is found in terms of known variables, we take the square root to find length c.

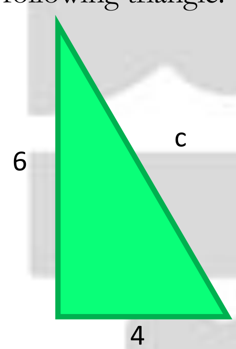
This process can be seen as:



This is the idea behind how the Pythagorean Theorem works. We can now take a look at how the theorem can be used to solve a problem.

Example 3

Problem: Find the length c for the following triangle:



Solution: We first recognize that this is a right triangle, and write our equation for the Pythagorean Theorem,

$$a^2 + b^2 = c^2.$$

We can choose either side of the triangle to be $a$or b. We enter the values into the equation and compute.

$$4^2 + 6^2 = c^2$$
$$(4 \times 4) + (6 \times 6) = c^2$$
$$16 + 36 = c^2$$
$$52 = c^2$$

The area of the square formed by our hypotenuse is 52. To return to our desired value c, we take the square root.

$$c^2 = 52$$
$$c = \sqrt{52}$$

We have now found the length of our hypotenuse. However, using the tree method will let us rewrite our answer in a preferable form.

$$52$$
$$26 \times 2$$
$$13 \times 2$$

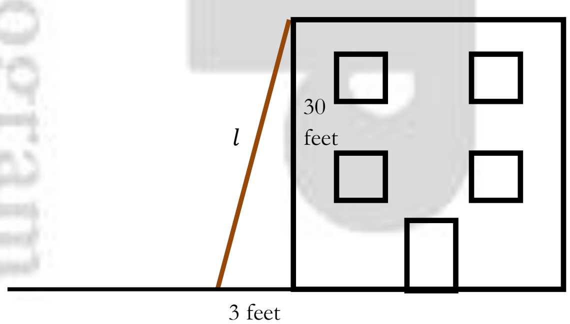
We recognize the square product within the number 52. This can be brought out from underneath the square root.

$$c = \sqrt{13 \times 2 \times 2}$$
$$c = \sqrt{13 \times 2^2}$$
$$c = \sqrt{13} \times \sqrt{2^2}$$
$$c = \sqrt{13} \times 2$$
$$c = 2\sqrt{13}$$

Example 4:

Problem: A ladder is being used to reach the top of a 30 foot building. To ensure stability, the base of the ladder is placed 3 feet from the wall. What is the minimal length the ladder must be to reach the top of the building?

Solution: Word problems such as this first require us to read the problem and draw a picture.



We see that this problem is an application of the Pythagorean Theorem. The procedure to find our unknown length l is the same as in example 3.

$$a^2 + b^2 = c^2$$
$$(3ft)^2 + (30ft)^2 = l^2$$
$$9ft^2 + 900ft^2 = l^2$$
$$909ft^2 = l^2$$
$$l = \sqrt{909ft^2}$$
$$l = \sqrt{909}ft$$

$$909$$
$$303 \times 3$$
$$101 \times 3$$

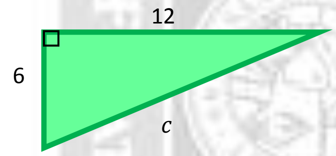
$$909 = 101 \times 3 \times 3$$
$$l = \sqrt{101 \times 3 \times 3}ft$$
$$l = \sqrt{101 \times 3^2}ft$$
$$l = \sqrt{101} \times \sqrt{3^2}ft$$
$$l = 3\sqrt{101}ft$$

The minimal length needed for our ladder is therefore $3\sqrt{101}$ feet, or approximately 30.15 feet. Notice how we included the units of feet when evaluating the problem. This helps give our final answer a physical meaning, since the ladder is measured in feet.

Consider: If the ladder's base were further from the building, how would the minimal length for the ladder change? Would our answer increase or decrease?

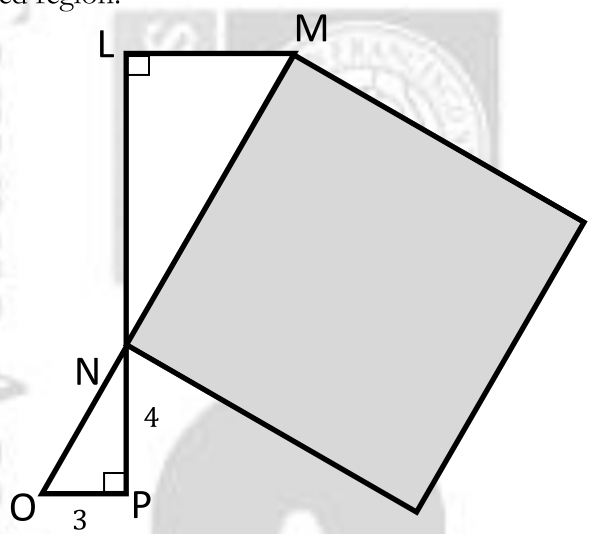
Practice Problems

1. Find the unknown side c for the following triangle.



2. Jane and Margaret are standing on opposite sides of a river, and Jane wants to know her distance from Margaret. 5 feet downstream from Jane is a large willow tree that lies directly across the river from Margaret. If the river is 20 feet wide, what is Jane's distance to Margaret?

3. Find the area of the shaded region:



Where $\overline{LM}$ is twice the length of $\overline{OP}$.