Campus Academic Resource Program
Differentiation Rules

This handout **will**:
- Outline the definition of the derivative and introduce different notations for the derivative
- Introduce Differentiation rules and provide concise explanation of them
- Provide examples of applications for Differentiation rules

This handout **will not** cover:
- The Chain Rule (The link to the handout on chain rule)
- Trigonometric Rule
- Logarithm Rule

Overview of the Derivative and Derivative notations

(For the information on how the derivative is defined in terms of the limit, refer to the limit definition of the derivative handout: http://carp.sfsu.edu/sites/sites7.sfsu.edu.carp/files/The Limit Definition of the Derivative.pdf)

This section will:
- Define a derivative
- Provide different notations for derivatives

Derivative is defined as the instantaneous rate of change of a function at a point. The derivative of $f(x_0)$ is:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

where h is the change between the initial value of x_0 and final value of $x_0 + h$. The definition of the derivative may be thought of as an extension of the slope formula (or "average rate of change" formula):

$$\frac{\Delta y}{\Delta x} = \frac{f(x)}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

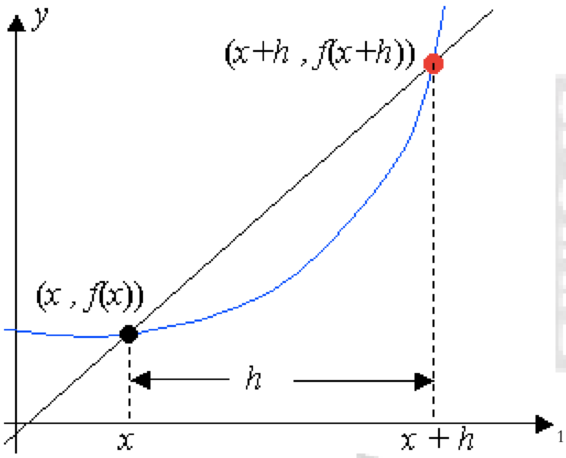
If we designate $x_1 = x_0 + h$ where h represents the change between x_1 and x_0, then:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0} = \frac{f(x_0 + h) - f(x_0)}{h}$$

Graphically, this equation is the line (known as the "secant line") that connects two points of the function, as seen in the figure at the top of the following page:

Campus Academic Resource Program

Differentiation Rules



[1]

The limit as h approaches 0 returns the derivative at $(f(x_0), x_0)$:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

There are several notations that may be used to represent differentiation:

$$\frac{dy}{dx} \text{ or } \frac{d(f(x)}{dx} \text{ or } \frac{d}{dx} f(x) \text{ or } f'(x)$$

$\frac{d}{dx}$ represents the process of differentiation[2] of the function $f(x)$. $f'(x)$ is another way of representing differentiation and is referred to as "the prime notation".

Information on how to calculate the derivative using the definition may be found here:
http://carp.sfsu.edu/sites/sites7.sfsu.edu.carp/files/The Limit Definition of the Derivative.pdf

In practice, the definition of the derivative is rarely used to find the derivative. It is more convenient to use Derivative Rules to find the derivative. The remainder of this handout is dedicated to introducing some of these derivative rules.

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[1] Image source: http://www.teacherschoice.com.au/first_principles.htm

[2] Differentiation is the operation of taking the derivative. We say that a function is "differentiable" at a point if "it has the derivative at that point". Accordingly, "to differentiate the function" is "to find the derivative of the function". "Derivative" corresponds to the verb "differentiate", not "derive".

Differentiation Rules
This section will:
- Introduce the following differentiation rules:
  - Derivative of a Constant Function
  - Power Rule
  - Constant Multiple Derivative Rule
  - Derivative Sum Rule
  - Product Rule
  - Quotient Rule
  - Derivative of the Natural Exponential Function
- Provide examples of applying these rules for finding derivatives
- Provide examples where some of these rules must be combined in order to find the derivative of a function.

Differentiation rules are efficient methods of finding the derivative without using the definition of derivative. Depending on the kind of function we are given, specific differentiation rules must be applied in order to find the derivative.

Derivative of a Constant Function:

Let c be a constant. The derivative of c is:

$$\frac{d}{dx}c = 0$$

To understand this rule intuitively, recall that derivatives measure the instantaneous ***rate of change*** of a function at a point. By definition, an output of a constant function does not change, and so its instantaneous rate of change is always zero. For example, suppose we are given the function

$$f(x) = 5$$

then the derivative of that function with respect to x is

$$\frac{d}{dx}f(x) = 0$$

Since the function is constant the instantaneous rate of change (i.e. the derivative) is 0.

Suppose we are given the function

$$f(x) = \sqrt[3]{\left(\pi - \frac{1}{2}\right)^2}$$

$\sqrt[3]{\left(\pi - \frac{1}{2}\right)^2}$ is a constant function as well (there are no variables present here, only numbers and constants), and so its derivative is 0:

$$\frac{d}{dx}\sqrt[3]{\left(\pi - \frac{1}{2}\right)^2} = 0$$

Power Rule

The **Power Rule** assists us in finding the derivative of a function where x is raised to some constant power n. The general form of the Power Rule is the following:

$$\frac{d}{dx}x^n = nx^{n-1}$$

For example, suppose we are asked to find the derivative of the function below:

$$f(x) = x^3$$

The function $f(x) = x^3$ is of the form x^n, where n is a constant, which means that we can find the derivative using the Power Rule:

$$\frac{d}{dx}(x^3) = 3x^{3-1}$$

$$= 3x^2$$

As another example, suppose we are asked to find the derivative of the function given below:

$$f(x) = \sqrt{x}$$

We may rewrite the function in the form of x^n where n is a fractional power[3]:

$$f(x) = x^{\frac{1}{2}}$$

We can use the Power Rule to find the derivative:

$$\frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2}x^{\left(\frac{1}{2}-1\right)}$$

$$= \frac{1}{2}x^{-\frac{1}{2}}$$

[3] All expressions of the kind $\sqrt[n]{(x)^c}$ can be represented as $x^{\frac{c}{n}}$ (note $\sqrt{x} \equiv \sqrt[2]{x}$) For example, $\sqrt[2]{x^1}$, which is generally represented as "the square root of x" or $\sqrt{x}$, is equal to $x^{\frac{1}{2}}$. Similarly, $\sqrt[3]{x^2} = x^{\frac{2}{3}}$.

Suppose we are given the following function:

$$f(x) = x^{\frac{\sqrt{\pi - \frac{3}{4}}}{7}}$$

Even though x raised to a power which is not as an integer, $\frac{\sqrt{\pi-\frac{3}{4}}}{7}$ is still *some* number. Therefore, we can apply the Power Rule here as well:

$$\frac{d}{dx} f(x) = \left(\frac{\sqrt{\pi - \frac{3}{4}}}{7}\right) x^{\left(\frac{\sqrt{\pi-\frac{3}{4}}}{7} - 1\right)}$$

Constant Multiple Rule

The Constant Multiple Rule tells us that if our function u contains some constant multiple c, then c can be "taken out" of the derivative:

$$\frac{d}{dx}(cu) = c\frac{d}{dx}(u)$$

For example, we are asked to find the derivative of the following function:

$$f(x) = 3x^2$$

To find the derivative, we will apply both the Constant Multiple Rule and the Power Rule:

$$\frac{d}{dx}(3x^2) = 3\frac{d}{dx}(x^2)$$

$$= 3(2)x^{2-1}$$

$$= 6x$$

Suppose we are asked to find the derivative of the following function:

$$f(x) = \left(\frac{\pi - 2}{n}\right) x^n$$

where n and π are constants. $\left(\frac{\pi-2}{n}\right)$ is some constant since it contains only constants. We apply the constant multiple rule to take $\left(\frac{\pi-2}{n}\right)$ out of derivative, and then find the derivative of x^n using the power rule:

$$\frac{d}{dx}\left(\left(\frac{\pi - 2}{n}\right) x^n\right) = \left(\frac{\pi - 2}{n}\right)\frac{d}{dx}(x^n)$$

$$= \left(\frac{\pi - 2}{n}\right) n x^{n-1}$$

$$= (\pi - 2)x^{n-1}$$

Derivative Sum Rule and Difference Rule

The **Derivative Sum Rule** says that the derivative of $u + v$ is identical to the derivative of u plus the derivative of v, symbolically[4].

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

For example, suppose we need to find the derivative of the following function:

$$f(x) = x^3 + 34$$

we can think of this function as the combination of the two following functions: $u(x) = x^3$ and $v(x) = 34$

$$f(x) = u(x) + v(x)$$

$$= x^3 + 34$$

We can now use the Derivative Sum Rule:

$$\frac{d}{dx}(x^3 + 34) = \frac{d}{dx}(x^3) + \frac{d}{dx}(34)$$

[4] An additional condition is u and v are differentiable (i.e. their derivative exists). We will assume all functions are differentiable unless otherwise stated.

Now we can use the Power Rule and the Derivative of a Constant Function Rule to find the derivative of $\frac{d}{dx}x^3$ and $\frac{d}{dx}34$

$$\frac{d}{dx}(x^3+34) = \frac{d}{dx}(x^3) + \frac{d}{dx}(34)$$

$$= 3x^2 + 0$$

$$= 3x^2$$

The Difference Rule is obtained from the Derivative Sum Rule:

$$\frac{d}{dx}(u-v) = \frac{d}{dx}\big(u+(-v)\big)$$

$$= \frac{du}{dx} + \frac{d(-v)}{dx}$$

$$= \frac{du}{dx} - \frac{dv}{dx}$$

Derivative of the Natural Exponential Function

The **Derivative of the Natural Exponential Function** (the function $f(x) = e^x$) rule helps us find the derivative of e^x without using of the definition of the derivative. Note that e is a constant; the value of e up to 2 decimal places is: $e \approx 2.72$. The derivative of e^x is:

$$\frac{d}{dx}(e^x) = e^x$$

Suppose we are given the function:

$$f(x) = 2e^x$$

We can use the Constant Multiple Rule and the derivative of e^x rule to find the derivative of $f(x)$

$$\frac{d}{dx}(2e^x) = 2\frac{d}{dx}e^x$$

$$= 2e^x$$

Product Rule

The **Product Rule** says the derivative of uv is u times the derivative of v plus v times the derivative of u. Symbolically:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

We use the Product Rule whenever we have the product of two functions that are differentiable with respect to x.

For example, suppose we want to find the derivative of the function:

$$f(x) = (x^2)(3x+1).$$

The function is a product of two functions: $u(x) = x^2$ and $v(x) = 3x + 1$. Applying the Product Rule:

$$\frac{d}{dx}[(x^2)(3x+1)] = x^2\left[\frac{d}{dx}(3x+1)\right] + \left[\frac{d}{dx}(x^2)\right](3x+1)$$

next, we apply the Power Rule and the Sum Rule to solve for the derivative of $v(x) = 3x + 1$ and $u(x) = x^2$:

$$x^2\left[\frac{d}{dx}(3x+1)\right] + \left[\frac{d}{dx}(x^2)\right](3x+1) = 3x^2 + 2x(3x+1)$$

$$= 3x^2 + 6x^2 + 2x$$

$$= 9x^2 + 2x$$

We can verify that our result is correct by expanding the function $f(x) = (x^2)(3x+1)$ and solving for the derivative using the Sum Rule and the Power Rule:

$$f(x) = (x^2)(3x+1)$$

$$= 3x^3 + x^2$$

$$\frac{d}{dx}(3x^3 + x^2) = 9x^2 + 2x$$

As we can see, both results agree.

However, not all functions can be simplified in order to avoid using the Power Rule. For example:

$$f(x) = 2xe^x$$

We must use the Power Rule to differentiate this function since the function cannot be simplified any further.

$$u(x) = 2x \quad v(x) = e^x$$

$$\frac{d}{dx}(2xe^x) = 2x\left[\frac{d}{dx}(e^x)\right] + \left[\frac{d}{dx}(2x)\right]e^x$$

To solve for the derivative, we use the Derivative of the Natural Exponential Function Rule and the Power Rule to find the derivatives of $v(x) = e^x$ and $u(x) = 2x$

$$2x\left[\frac{d}{dx}(e^x)\right] + \left[\frac{d}{dx}(2x)\right]e^x = 2xe^x + 2e^x$$

Quotient Rule

Suppose we have a function $f(x)$ that can be decomposed into the quotient of two simpler functions in the form of $f(x) = \frac{u(x)}{v(x)}$. To find the derivative of $f(x)$ we use the following equation

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{u(x)}{v(x)}\right)$$

$$= \frac{\left[\frac{d}{dx}u(x)\right]v(x) - u(x)\left[\frac{d}{dx}v(x)\right]}{[v(x)]^2}$$

or in prime notation

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

The **Quotient Rule** is useful for finding the derivative of a function that is presented in the fractional form. Consider the following function

$$f(x) = \frac{x^2 - 2}{x^3}$$

Since this function is given in the fractional form we can apply the Quotient Rule to find the derivative of the function. We can think of the numerator as a function $u(x) = x^2 - 2$ and the denominator as $v(x) = x^3$. Applying the Quotient Rule:

$$\frac{d}{dx}\left(\frac{x^2 - 2}{x^3}\right) = \frac{\left[\frac{d}{dx}(x^2 - 2)\right](x^3) - (x^2 - 2)\left[\frac{d}{dx}(x^3)\right]}{(x^3)^2}$$

From there, we apply the Power Rule and the Sum Rule to find the derivatives:

$$\frac{\left[\frac{d}{dx}(x^2-2)\right](x^3)-(x^2-2)\left[\frac{d}{dx}(x^3)\right]}{(x^3)^2}=\frac{(2x)(x^3)-(x^2-2)(3x^2)}{x^6}$$
$$=\frac{6x^2-x^4}{x^6}$$
$$=\frac{6-x^2}{x^4}$$

Conclusion:

In this handout we have done the following:

- Reviewed the definition and notations of the derivative
- Explored differentiation rules
- Provided some examples of application of these rules, and also examples where we had to use several rules to find the derivative. See the following page for practice problems.
- Provided a list of derivative rules (see the last page of the document).

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Differentiation Rules

Practice problem: Find derivatives of the following functions (answers are on the next page):

1. $f(x) = 5x^{\frac{3}{5}} - 47$

2. $f(x) = (x + 5)(x^2 - 4x)$

3. $f(x) = 3xe^x + \frac{\sqrt{x^3}}{e^x}$

Campus Academic Resource Program
Differentiation Rules

Answers to Practice problems:

1) $f'(x) = 3x^{-\frac{2}{5}}$

2) $f'(x) = 3x^2 + 2x - 20$

3) $f'(x) = 3e^x + 3xe^x - e^{-x}\sqrt{x^3} + \frac{3}{2}e^{-x}\sqrt{x}$

Reference:
George B. Thomas, Jr "Thomas' Calculus Early Transcendentals" Twelfth Edition, 2010

Campus Academic Resource Program
Differentiation Rules

Summary of Differentiation rules:

Derivative of Constant Function	$\frac{d}{dx}(c) = 0$ (Where c is a constant)
Power Rule	$\frac{d}{dx}(x^n) = nx^{n-1}$ (Where n is a constant)
Constant Multiple Rule	$\frac{d}{dx}(cx) = c\frac{d}{dx}(x)$ (where c is a constant)
Derivative Sum Rule/Difference Rule	$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$ $\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$
Derivative of Natural Exponential Function	$\frac{d}{dx}(e^x) = e^x$
Product Rule	$\frac{d}{dx}(uv) = \frac{d}{dx}(u)(v) + (u)\frac{d}{dx}(v)$
Quotient Rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{d}{dx}(u)(v) - (u)\frac{d}{dx}(v)}{v^2}$